6.3/6.4 (~~&~~ 6.5)

Trigonometric Substitution

form	substitution
$\sqrt{b^2x^2 - a^2}$	$x = \frac{a}{b}\sec\theta$
$\sqrt{a^2 - b^2x^2}$	$x = \frac{a}{b}\sin\theta$
$\sqrt{a^2 + b^2x^2}$	$x = \frac{a}{b}\tan\theta$

- ★ Pythagorean Identities
- ★ Double & Half Angle Identities
- ★ $\int \sec x\, dx$ (mult by $\frac{\tan x + \sec x}{\tan x + \sec x}$)

Ex #1 $\int \frac{\sqrt{25x^2-4}}{x}\,dx = \int \frac{\sqrt{25(\frac{4}{25}\sec^2\theta) - 4}}{\frac{2}{5}\sec\theta} \cdot \frac{2}{5}\sec\theta\tan\theta\, d\theta$

$x = \frac{2}{5}\sec\theta$

$dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta$

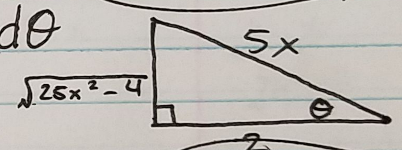
$= 2\int \sqrt{\sec^2\theta - 1}\ \tan\theta\, d\theta$

$\frac{5x}{2} = \sec\theta$

$\theta = \sec^{-1}\left(\frac{5x}{2}\right)$

$= 2\int \sqrt{\tan^2\theta}\ \tan\theta\, d\theta$

$= 2\int \tan\theta\ \tan\theta\, d\theta$



$= 2\int \tan^2\theta\, d\theta$

$= 2\int \sec^2\theta - 1\, d\theta$

$= 2(\tan\theta - \theta) + C$

$\tan\theta = \frac{\sqrt{25x^2-4}}{2}$

$= 2\left(\frac{\sqrt{25x^2-4}}{2} - \sec^{-1}\left(\frac{5x}{2}\right)\right) + C$

$= \sqrt{25x^2-4} - 2\sec^{-1}\left(\frac{5x}{2}\right) + C$

~~Ex #2 $\int \frac{1}{x^4\sqrt{9-x^2}}\,dx$~~

$\longrightarrow$ Next Page!

Ex #2 $\displaystyle\int \frac{1}{x^4\sqrt{9-x^2}}\,dx = \int \frac{3\cos\theta}{81\sin^4\theta\sqrt{9-9\sin^2\theta}}\,d\theta$

$x = 3\sin\theta$
$dx = 3\cos\theta\, d\theta$

$$= \int \frac{\cancel{3}\cancel{\cos\theta}\, d\theta}{81\sin^4\theta\, \cancel{3}\sqrt{\cancel{\cos^2\theta}}}$$

$$= \frac{1}{81}\int \frac{d\theta}{\sin^4\theta}$$

$$= \frac{1}{81}\int \csc^4\theta\, d\theta$$

$$= \frac{1}{81}\int \csc^2\theta\csc^2\theta\, d\theta$$

$$= \frac{1}{81}\int \csc^2\theta(\cot^2\theta+1)\, d\theta$$

$u = \cot\theta$
$du = -\csc^2\theta\, d\theta$
$-du = \csc^2\theta\, d\theta$

$$= -\frac{1}{81}\int u^2 + 1\, du$$

$$= -\frac{1}{81}\left(\frac{u^3}{3} + u\right) + C$$

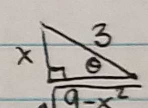
$$= -\frac{1}{81}\left(\frac{1}{3}\cot^3\theta + \cot\theta\right) + C$$

$$= -\frac{1}{81}\left(\frac{1}{3}\left(\frac{\sqrt{9-x^2}}{x}\right)^3 + \frac{\sqrt{9-x^2}}{x}\right) + C$$

$x = 3\sin\theta$

$$= -\frac{(9-x^2)^{3/2}}{243x^3} - \frac{\sqrt{9-x^2}}{81x} + C$$

$\sin\theta = \frac{x}{3}$



L'Hôpital's

$\lim_{x\to a} f(x)/g(x) = \frac{0}{0}$ or $\lim_{x\to a} f(x)/g(x) = \frac{\infty}{\infty}$

These are equivalent to $\lim_{x\to a} f'(x)/g'(x) =$

Ex #3 $\displaystyle\lim_{x\to\infty}\frac{(x+5)^2}{e^{3x}} = \lim_{x\to\infty}\frac{2(x+5)}{3e^{3x}}$

$$= \lim_{x\to\infty}\frac{2}{9e^{3x}}$$

$$= 0$$

Ex #4 $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^x}$

$$= \lim_{x\to\infty} e^{x\ln\frac{x+1}{x}}$$

$$= e^{\lim_{x\to\infty}\frac{\ln\frac{x+1}{x}}{\frac{1}{x}}}$$

* this is the step where we may apply L'Hôpital's Rule

$$= e^{\lim_{x\to\infty}\frac{\frac{x}{x+1}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}}$$

$$= e^{\lim_{x\to\infty}\frac{x}{x+1}}$$

$$= e^{\lim_{x\to\infty}\frac{1}{1}}$$

$$= e^1$$

$$= e$$