7.8

<u>Taylor Series</u> (for $f(x)$ about $x=a$)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \frac{f(a)}{0!}(x-a)^0 + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2$$

$$+ \frac{f'''(a)}{3!}(x-a)^3 + \cdots +$$

A Maclaurin Series is a Taylor Series with $a=0$.

Ex#1 Write a Taylor Series expansion for $f(x)=\ln x$ about $a=2$.

$$f(x) = \frac{\ln a}{0!}(x-a)^0 + \frac{\frac{1}{a}}{1!}(x-a)^1 + \frac{-\frac{1}{a^2}}{2!}(x-a)^2$$

$$+ \frac{\frac{2}{a^3}}{3!}(x-a)^3 + \frac{-\frac{3\cdot 2}{a^4}}{4!}(x-a)^4 + \frac{\frac{4\cdot 3\cdot 2}{a^5}}{5!}(x-a)^5 + \cdots$$

$$\boxed{f(x) = \ln 2 + \tfrac{1}{2}(x-2) - \tfrac{1}{8}(x-2)^2 + \tfrac{1}{24}(x-2)^3 - \tfrac{1}{64}(x-2)^4 + \tfrac{1}{160}(x-2)^5 + \cdots}$$

Ex#2 Write a Taylor Series for ex#1 using sigma notation.

$$a_n = (-1)^{n+1}\frac{1}{a^n}\left(\frac{1}{n}\right)(x-a)^n$$

$$\boxed{f(x) = \ln 2 + \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{2^n n}\right)(x-2)^n}$$

Ex#3 Find the Taylor Series expansion for $f(x)=\frac{1}{x^2}$ about $a=-1$.

$$\boxed{f(x) = 1 + 2(x+1) + 3(x+1)^2 + 4(x+1)^3 + 5(x+1)^4 + 6(x+1)^5 \ldots}$$

Ex#4 ↗ w/ sigma notation.

$$a_n = (n+1)(x+1)^n$$

$$\boxed{f(x) = \sum_{n=0}^{\infty}(n+1)(x+1)^n}$$

Lagrange Error Bound (AKA Remainder/Error Function)

Let $T_n(x)$ be the sum of the first n terms of $f(x)$ about $x=a$.

The error by approximating the infinite sum using $T_n(x)$ is determind by finding the Lagrange Error Bound.

$$\text{Error} = E_n(x) = R_n(x)$$

$$R_n(x) = \underbrace{\sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i}_{} - \overbrace{\sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i}^{T_n(x)}$$

⋮ math :)

$$R_n(x) \le \left| \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \right|$$

given: n, a, x

c will be the x-value between x & a (might be x or a) that maximizes the abs. val. of $(n+1)^{th}$ derivative function.

Ex#5 Determine the maximum error in using the first five terms $(n=4)$ of the Taylor Series representing $f(x)=\cos x$ about $a=0$ at $x=\pi/4$.

$n=4$

$a=0$

$x=\pi/4$

$c=\pi/4$

$f^{(5)} = -\sin c$

* c = x-value that will maximize $-\sin c$ between 0 & $\pi/4$ *

$f^{(5)} = -\sin \pi/4 = -1/\sqrt{2}$

$$R_4\left(\frac{\pi}{4}\right) \le \left| \frac{-\frac{1}{\sqrt{2}}}{5!}\left(\frac{\pi}{4}-0\right)^5 \right|$$

$$R_4(\pi/4) \le 1.76 \times 10^{-3}$$

$$= 0.00176$$