5.3 Sum & Difference Identities

Sum & Differences

Cosine: $\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$

Sine: $\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$

Tangent: $\tan(u \pm v) = \frac{\sin(u \pm v)}{\cos(u \pm v)} = \frac{\sin u \cos v \pm \cos u \sin v}{\cos u \cos v \mp \sin u \sin v}$

$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$

Ex #1 Determine the exact value of $\cos 75°$ without a calculator.

$$\cos 75° = \cos(45° + 30°)$$
$$= \cos 45° \cos 30° - \sin 45° \sin 30°$$
$$= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$
$$\boxed{\cos 75° = \frac{\sqrt{6} - \sqrt{2}}{4}}$$

Ex #2 Evaluate $\sin(165°)$ without a calculator.

$$\sin(165°) = \sin(120° + 45°)$$
$$= \sin 120° \cos 45° + \cos 120° \sin 45°$$
$$= \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$
$$\boxed{\sin(165°) = \frac{\sqrt{6} - \sqrt{2}}{4}}$$

Ex #3 Write as the cosine of an angle.

$$\boxed{\cos x \cos \frac{\pi}{7} - \sin x \sin \frac{\pi}{7} = \cos\left(x + \frac{\pi}{7}\right)}$$

Ex #4 Prove $\sin\left(\frac{\pi}{2} - x\right) = \cos x$ using sum/diff id.

$$\sin\left(\frac{\pi}{2} - x\right) = \sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x$$
$$= (1)\cos x - 0 \sin x$$
$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

Ex #5 Prove $\sin(x+\pi) = -\sin x$.

$$\sin(x+\pi) = \sin x \cos\pi + \cos x \sin\pi$$
$$= (\sin x)(-1) + (\cos x)(0)$$
$$\sin(x+\pi) = -\sin x \quad \blacksquare$$

Ex #6 Prove $\tan\left(\theta - \frac{3\pi}{2}\right) = -\cot\theta$.

$$\tan\left(\theta - \frac{3\pi}{2}\right) = \frac{\sin\left(\theta - \frac{3\pi}{2}\right)}{\cos\left(\theta - \frac{3\pi}{2}\right)}$$
$$= \frac{\sin\theta\cos\frac{3\pi}{2} - \cos\theta\sin\frac{3\pi}{2}}{\cos\theta\cos\frac{3\pi}{2} + \sin\theta\sin\frac{3\pi}{2}}$$
$$= \frac{(\sin\theta)(0) - (\cos\theta)(-1)}{(\cos\theta)(0) + (\sin\theta)(-1)}$$
$$= \frac{\cos\theta}{-\sin\theta}$$
$$\tan\left(\theta - \frac{3\pi}{2}\right) = -\cot\theta \quad \blacksquare \quad \text{q.ed.}$$

Ex #7 Prove $\sin 3x = 3\cos^2 x \sin x - \sin^3 x$.

$$\sin 3x = \sin(2x + x)$$
$$= \sin 2x \cos x + \cos 2x \sin x$$
$$= \sin(x+x)\cos x + \cos(x+x)\sin x$$
$$= [\sin x\cos x + \cos x \sin x]\cos x + [\cos x\cos x - \sin x \sin x]\sin x$$
$$= \sin x\cos^2 x + \cos^2 x \sin x + \cos^2 x \sin x - \sin^3 x$$
$$\sin 3x = 3\cos^2 x \sin x - \sin^3 x \quad \blacksquare$$