2.4 Real Zeros of Polynomial Functions

Long Division

Polynomial long division always works, BUT you must remember to add zero for missing terms and write the polynomial in standard form.

Ex#1 Divide $3x^3 - 5x^2 + 10x - 3$ by $3x+1$.

$$x^2 - 2x + 4 + \frac{-7}{3x+1}$$

$$3x+1 \overline{)\,3x^3 - 5x^2 + 10x - 3}$$

$$-(3x^3 + x^2)$$

$$-6x^2 + 10x$$

$$-(-6x^2 - 2x)$$

$$12x - 3$$

$$-(12x + 4)$$

$$-7$$

$$\frac{3x^3 - 5x^2 + 10x - 3}{3x+1}$$

$$= x^2 - 2x + 4 - \frac{7}{3x+1} \quad \text{Fractional}$$

$$= (x^2 - 2x + 4)(3x+1) - 7 \quad \text{Polynomial}$$

Ex#2 $(2x^3 - 9x^2 + 15)/(2x-5)$

Quotient $q(x)$

$$x^2 - 2x - 5 - \frac{10}{2x-5}$$

Remainder $r(x)$

Divisor $d(x)$

$$2x-5 \overline{)\,2x^3 - 9x^2 + 0x + 15}$$

Dividend $f(x)$

$$-(2x^3 - 5x^2)$$

$$-4x^2 + 0x$$

$$-(-4x^2 + 10x)$$

$$-10x + 15$$

$$-(-10x + 25)$$

$$-10$$

$$\boxed{\begin{aligned} \frac{2x^3 - 9x^2 + 15}{2x-5} &= x^2 - 2x - 5 - \frac{10}{2x-5} \\ &= (x^2 - 2x - 5)(2x-5) - 10 \end{aligned}}$$

Synthetic Division

Only work for divisors in the form $x-k$.

Ex #3 Divide $2x^3-5x^2+3x+7$ by $1x-2$.

$x-2=0$
$x=2$

$$\begin{array}{r|rrrr} 2 & 2 & -5 & 3 & 7 \\ & & 4 & -2 & 2 \\ \hline & 2 & -1 & 1 & \vert\, 9 \\ & x^2 & x & c & r(x) \end{array}$$

$$\boxed{\frac{2x^3-5x^2+3x+7}{x-2} = 2x^2-x+1+\frac{9}{x-2} = (2x^2-x+1)(x-2)+9}$$

No remainder for long/synthetic division means $d(x)$ is a factor of $f(x)$.

Factor Theorem

A polynomial function $f(x)$ has a factor $(x-k)$ iff $f(k)=0$.

Remainder Theorem

If a polynomial $f(x)$ is divided by $(x-k)$, then the remainder is $r=f(k)$.

2.4 con't

Rational Root Theorem

If $P(x)$ is a polynomial with integer coefficients and if $\frac{p}{q}$ is a zero of $P(x)$ $\left(P\left(\frac{p}{q}\right)=0\right)$, then p is a factor of the constant term and q is a factor of the leading coefficient.

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

q is a factor of a_n — p is a factor of a_0

1. List all values of p
2. List all values of q
3. List all values of p/q
4. Use division to determine the values of p/q for which $P(p/q)=0$.

NOTE: This gives you the possible rational roots

Ex #4 Find all the rational zeros of

$$P(x) = x^3 - 9x + 9 + 2x^4 - 19x^2$$

$$P(x) = 2x^4 + x^3 - 19x^2 - 9x + 9$$

$p = \pm 1, \pm 9, \pm 3$

$q = \pm 1, \pm 2$

$\frac{p}{q} = \pm 1, \pm 9, \pm 3, \pm\frac{1}{2}, \pm\frac{9}{2}, \pm\frac{3}{2}$ possible rational roots

$$\begin{array}{r|rrrrr} 1 & 2 & 1 & -19 & -9 & 9 \\ & & 2 & 3 & -16 & -25 \\ \hline & 2 & 3 & -16 & -25 & \cancel{\;} \end{array} \quad (x-1)$$

$$\begin{array}{r|rrrrr} -1 & 2 & 1 & -19 & -9 & 9 \\ & & -2 & 1 & 18 & -9 \\ \hline & 2 & -1 & -18 & 9 & 0 \end{array}$$

$P(x) = (x+1)(2x^3 - x^2 - 18x + 9)$

$$\begin{array}{r|rrrr} 3 & 2 & -1 & -18 & 9 \\ & & 6 & 15 & -9 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

$P(x) = (x+1)(x-3)(2x^2+5x-3)$

$$\begin{array}{r|rrr} -3 & 2 & 5 & -3 \\ & & -6 & 3 \\ \hline & 2 & -1 & 0 \end{array}$$

$P(x) = (x+1)(x-3)(x+3)(2x-1)$

$\boxed{\text{Rational zeros: } -1, 3, -3, \frac{1}{2}}$

Ex#5 $f(x)=3x^3+4x^2-5x-2$ Find all rat'l roots.

$P \Rightarrow \pm(1,2)$ $\qquad \frac{p}{q}=\pm\left(1,2,\frac{1}{3},\frac{2}{3}\right)$

$q \Rightarrow \pm(1,3)$

$$\begin{array}{r|rrrr} -1 & 3 & 4 & -5 & -2 \\ & & -3 & -1 & 6 \\ \hline & 3 & 1 & -6 & 4 \end{array}$$

$f(x)=(x-1)(3x^2+7x+2)$

$$\begin{array}{r|rrrr} 1 & 3 & 4 & -5 & -2 \\ & & 3 & 7 & 2 \\ \hline & 3 & 7 & 2 & 0 \end{array}$$

$f(x)=(x-1)(x+2)(3x+1)$

$6 \quad 2 \quad \frac{6}{3} \quad \frac{1}{3} \quad 7$

$3x^2+6x+1x+2$

$3x(x+2)+1(x+2)$

$(x+2)(3x+1)$

$\boxed{\text{Rat'l zeros: } 1, -2, -\frac{1}{3}}$

Ex#6 $f(x)=2x^4-7x^3-8x^2+14x+8$ Find all zeros

$$\frac{p}{q}=\pm\frac{1,8,2,4}{1,2}$$

$$\begin{array}{r|rrrrr} 1 & 2 & -7 & -8 & 14 & 8 \\ & & 2 & -5 & -13 & 1 \\ \hline & 2 & -5 & -13 & 1 & 9 \end{array}$$

$$\begin{array}{r|rrrrr} -1 & 2 & -7 & -8 & 14 & 8 \\ & & -2 & 9 & -1 & -13 \\ \hline & 2 & -9 & 1 & 13 & -5 \end{array}$$

$$\begin{array}{r|rrrrr} 4 & 2 & -7 & -8 & 14 & 8 \\ & & 8 & 4 & -16 & -8 \\ \hline -\frac{1}{2} & 2 & 1 & -4 & -2 & 0 \\ & & -1 & 0 & 2 & \\ \hline & 2 & 0 & -4 & 0 & \end{array}$$

$(\cancel{2x+1})$ or $\left(x+\frac{1}{2}\right)$

$f(x)=(x-4)(\cancel{2x+1})(2x^2-4)\left(x+\frac{1}{2}\right)$

$0=2x^2-4=2(x^2-2)$

$0=x^2-2$

$2=x^2$

$\pm\sqrt{2}=x$

$\boxed{\text{zeros: } 4, -\frac{1}{2}, \pm\sqrt{2}}$