5.2 Proving Trigonometric Identities

General Strategy:

1) Begin with the more complicated expression on the LHS.
2) Arrive at the RHS by using a sequence of equivalent expressions, using one identity at a time.
3) If you're stuck, convert everything to sines & cosines.
4) Combine rational expressions w/ a common denominator.
5) Remember $(a+b)(a-b)=a^2-b^2$ for Pythagorean ID.
6) Keep the goal (RHS) in mind!

Ex #1 Prove $\cot x + \tan x = \sec x \csc x$.

$$\cot x + \tan x = \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}$$

$$= \frac{\cos^2 x}{\sin x \cos x} + \frac{\sin^2 x}{\cos x \sin x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\sin x \cos x}$$

$$= \frac{1}{\sin x \cos x}$$

$$= \csc x \sec x$$

$$\cot x + \tan x = \sec x \csc x \quad \blacksquare$$

Ex #2 Prove $\sin^4 x - \cos^4 x = 2\sin^2 x - 1$.

$$\begin{aligned}
\sin^4 x - \cos^4 x &= (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)\\
&= (\sin^2 x - \cos^2 x)(1)\\
&= \sin^2 x - (1 - \sin^2 x)\\
&= \sin^2 x - 1 + \sin^2 x\\
\sin^4 x - \cos^4 x &= 2\sin^2 x - 1 \quad \blacksquare
\end{aligned}$$

Ex #3 Prove $\dfrac{\cot^2 u}{1+\csc u} = (\cot u)(\sec u - \tan u)$.

$$\begin{aligned}
\frac{\cot^2 u}{1+\csc u} &= \frac{\csc^2 u - 1}{1+\csc u}\\
&= \frac{(\csc u - 1)\cancel{(\csc u + 1)}}{\cancel{1+\csc u}}\\
&= \csc u - 1\\
&= \frac{1}{\sin u} - 1
\end{aligned}$$

$$\begin{aligned}
(\cot u)(\sec u - \tan u) &= \left(\frac{\cos u}{\sin u}\right)\left(\frac{1}{\cos u} - \frac{\sin u}{\cos u}\right)\\
&= \left(\frac{\cancel{\cos u}}{\sin u}\right)\left(\frac{1-\sin}{\cancel{\cos u}}\right)\\
&= \frac{1-\sin u}{\sin u}\\
&= \frac{1}{\sin u} - 1\\
&= (\csc u - 1)\left(\frac{\csc u + 1}{1+\csc u}\right)\\
&= \frac{\csc^2 u - 1}{1+\csc u}\\
(\cot u)(\sec u - \tan u) &= \frac{\cot^2 u}{1+\csc u} \quad \blacksquare
\end{aligned}$$