7.14

Derivatives w/ Polar

$$\frac{dy}{dx} = \frac{\left(\frac{dr}{d\theta}\right)\sin\theta + (r)\cos\theta}{\left(\frac{dr}{d\theta}\right)\cos\theta - (r)\sin\theta}$$

Ex#1 Determine the equation of the tangent line to $r = 3 + 8\sin\theta$ @ $\theta = \pi/6$.

Point (x, y): $\theta = \pi/6$

$r = 3 + \frac{8}{2} = 7$

$x = r\cos\theta = \frac{7\sqrt{3}}{2}$

$y = r\sin\theta = 7/2$ $\quad\left(\frac{7\sqrt{3}}{2}, \frac{7}{2}\right)$

Slope: $\frac{dr}{d\theta} = 8\cos\theta$

$$\frac{dy}{dx} = \frac{(8\cos\theta)\sin\theta + (3+8\sin\theta)\cos\theta}{(8\cos\theta)\cos\theta - (3+8\sin\theta)\sin\theta}$$

$$\frac{dy}{dx} = \frac{8\cos\theta\sin\theta + 3\cos\theta + 8\cos\theta\sin\theta}{8\cos^2\theta - 3\sin\theta - 8\sin^2\theta}$$

$$\frac{dy}{dx} = \frac{16\cos\theta\sin\theta + 3\cos\theta}{8(\cos^2\theta - \sin^2\theta) - 3\sin\theta}$$

$$\frac{dy}{dx} = \frac{16\cos\theta\sin\theta + 3\cos\theta}{8\cos 2\theta - 3\sin\theta}$$

$$\frac{dy}{dx} = \frac{8\sin 2\theta + 3\cos\theta}{8\cos 2\theta - 3\sin\theta}$$

$7\pi/6$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \frac{11\sqrt{3}}{5}$$

$\pi/6$ $\quad 11\pi/6$

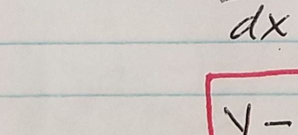
$7\pi/6$

$$\boxed{y - \frac{7}{2} = \frac{11\sqrt{3}}{5}\left(x - \frac{7\sqrt{3}}{2}\right)}$$

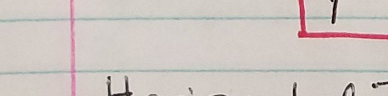
Horizontal Tangents: $0 = \frac{dr}{d\theta}\sin\theta + r\cos\theta$

Vertical Tangents: $0 = \frac{dr}{d\theta}\cos\theta - r\sin\theta$

Area Enclosed by Polar Curves

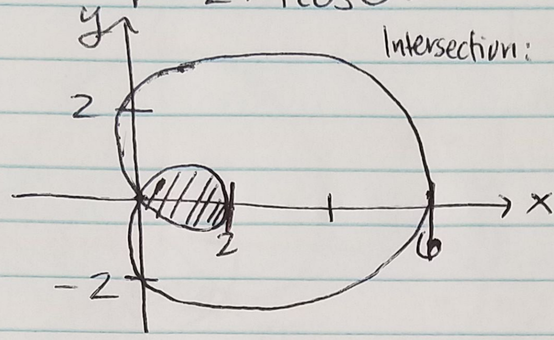


$$A = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta$$



$$A = \int_\alpha^\beta \frac{1}{2}\left(r_o^2 - r_i^2\right) d\theta$$

Ex#2 Determine the area of the inner loop of $r=2+4\cos\theta$.



Intersection: must figure out where the beginning & end of the loop is → origin

$0=2+4\cos\theta$

$-2/4=\cos\theta$

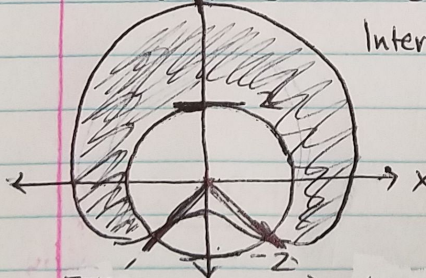
$-1/2=\cos\theta$

$\theta=\frac{2\pi}{3},\ \frac{4\pi}{3}$

$\alpha \nearrow$ $\quad \beta \nearrow$

$$A=\int_{2\pi/3}^{4\pi/3}\frac{1}{2}(2+4\cos\theta)^2\,d\theta$$
$$=\int_{2\pi/3}^{4\pi/3}\frac{1}{2}(4+16\cos\theta+16\cos^2\theta)\,d\theta$$
$$=\int_{2\pi/3}^{4\pi/3}2+8\cos\theta+8\cos^2\theta\,d\theta$$
$$=\int_{2\pi/3}^{4\pi/3}2+8\cos\theta+\frac{8}{2}(1+\cos2\theta)\,d\theta$$
$$=\int_{2\pi/3}^{4\pi/3}6+8\cos\theta+4\cos2\theta\,d\theta$$
$$=\left[6\theta+8\sin\theta+2\sin2\theta\right]_{2\pi/3}^{4\pi/3}$$
$$=8\pi-4\sqrt{3}+\sqrt{3}-(4\pi+4\sqrt{3}-\sqrt{3})$$
$$\boxed{A=4\pi-6\sqrt{3}}$$

Ex#3 Determine the area that lies inside $r=3+2\sin\theta$ & outside $r=2$.



$7\pi/6$ $\quad 11\pi/6,\ -\pi/6$

Intersection: $3+2\sin\theta=2$

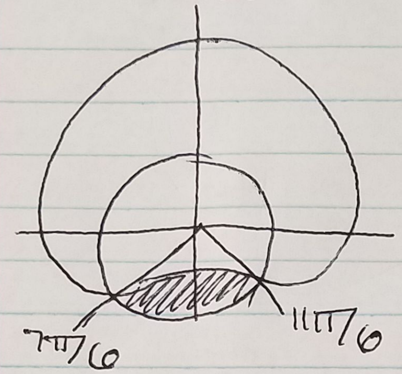
$\sin\theta=-1/2$

$\theta=\frac{7\pi}{6}$ & $\frac{11\pi}{6}$

$\alpha\nearrow$ $\quad \beta\nearrow$

$$A=\int_{-\pi/6}^{7\pi/6}\frac{1}{2}\left((3+2\sin\theta)^2-(2)^2\right)d\theta$$
$$=\int_{-\pi/6}^{7\pi/6}\frac{1}{2}(9+12\sin\theta+4\sin^2\theta-4)\,d\theta$$
$$=\int_{-\pi/6}^{7\pi/6}\frac{1}{2}\left(5+12\sin\theta+\frac{4}{2}(1-\cos2\theta)\right)d\theta$$
$$=\int_{-\pi/6}^{7\pi/6}\frac{7}{2}+6\sin\theta-\frac{2\cos2\theta}{2}\,d\theta$$
$$=\int \quad \vdots$$
$$=\left[\frac{7}{2}\theta-6\cos\theta-\frac{1}{2}\sin2\theta\right]$$
$$\boxed{A=11\sqrt{3}/2+14\pi/3}$$

Ex #4 Determine the area outside $r = 3 + 2\sin\theta$ & inside $r = 2$.



Intersection: same as ex #3

$$A = \int_{7\pi/6}^{11\pi/6} \frac{1}{2}\left(2^2 - (3+2\sin\theta)^2\right) d\theta$$

$$= \frac{1}{2}\int_{7\pi/6}^{11\pi/6} (4 - 9 - 12\sin\theta - 4\sin^2\theta)\, d\theta$$

$$= \int_{7\pi/6}^{11\pi/6} \frac{1}{2}\left(-5 - 12\sin\theta - \frac{4}{2}(1-\cos 2\theta)\right) d\theta$$

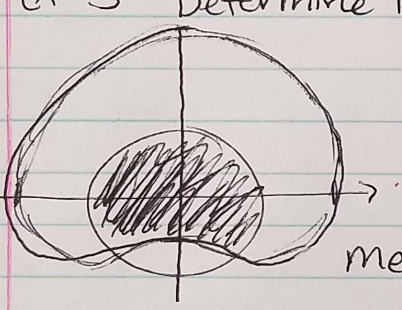
$$= \int_{7\pi/6}^{11\pi/6} \frac{1}{2}(-7 - 12\sin\theta + 2\cos 2\theta)\, d\theta$$

$$= \int_{7\pi/6}^{11\pi/6} \left(-\frac{7}{2} - 6\sin\theta + \cos 2\theta\right) d\theta$$

$$= \left[-\frac{7}{2}\theta + 6\cos\theta + \frac{1}{2}\sin 2\theta\right]_{7\pi/6}^{11\pi/6}$$

$$\boxed{A = \frac{11\sqrt{3}}{2} - \frac{7\pi}{3}}$$

Ex #5 Determine the area shared by $r = 3 + 2\sin\theta$ & $r = 2$.



method #1 : $A = A_{circle} - A_{ex\#4}$

$$A = \pi(2)^2 - \left(\frac{11\sqrt{3}}{2} - \frac{7\pi}{3}\right)$$

$$\boxed{A = \frac{19\pi}{3} - \frac{11\sqrt{3}}{2}}$$

method #2 : $A = A_{limaçon} - A_{ex\#3}$

$$A = \frac{1}{2}\int_0^{2\pi} (3+2\sin\theta)^2 d\theta - \left(\frac{11\sqrt{3}}{2} + \frac{14\pi}{3}\right)$$

$$\boxed{A = \frac{19\pi}{3} - \frac{11\sqrt{3}}{2}}$$

Arc Length in Polar

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Ex #6 Determine the length of $r = \theta$ for $0 \le \theta \le 1$.

$$L = \int_0^1 \sqrt{\theta^2 + 1^2}\, d\theta$$

$$= \int_0^{\pi/4} \sqrt{\tan^2 x + 1}\, \sec^2 x\, dx \qquad \begin{array}{l} \theta = \tan x \\ d\theta = \sec^2 x\, dx \end{array} \qquad \begin{array}{l} \theta = 1 \to x = \pi/4 \\ \theta = 0 \to x = 0 \end{array}$$

$$= \int_0^{\pi/4} \sec^3 x\, dx$$

$\vdots$

$$= \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right)\Big|_0^{\pi/4}$$

$$\boxed{L = \frac{1}{2}\left(\sqrt{2} + \ln(1+\sqrt{2})\right)}$$