7.4 Partial Fractions

Consider $\boxed{\dfrac{2}{x} + \dfrac{1}{x-2}} = \boxed{\dfrac{3x-4}{x^2-2x}} \longrightarrow$ rational expression

$\hookrightarrow$ partial fraction decomposition

Ex # 1  $\dfrac{3x+11}{x^2-x-6} = \dfrac{3x+11}{(x-3)(x+2)} = \dfrac{A}{x-3} + \dfrac{B}{x+2}$

$\left[ \dfrac{3x+11}{(x-3)(x+2)} = \dfrac{A}{x-3} + \dfrac{B}{x+2} \right] (x-3)(x+2)$

$3x+11 = A(x+2) + B(x-3)$

$3x+11 = Ax + 2A + Bx - 3B$

$3x+11 = Ax + Bx + 2A - 3B$

$3x+11 = x(A+B) + 2A - 3B$

$A + B = 3 \qquad\qquad 11 = 2A - 3B$

(Do substitution or elimination)

$A = 4 \qquad\qquad B = -1$

$$\boxed{\dfrac{3x+11}{x^2-x-6} = \dfrac{4}{x-3} + \dfrac{-1}{x+2}}$$

Ex #2  $\dfrac{x^2+4}{3x^3+4x^2-4x} = \dfrac{x^2+4}{x(3x^2+4x-4)} = \dfrac{x^2+4}{x(x+2)(3x-2)}$

$\dfrac{x^2+4}{x(x+2)(3x-2)} = \dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{3x-2}$

$x^2+4 = A(x+2)(3x-2) + B(x)(3x-2) + C(x)(x+2)$

$x^2+4 = A(3x^2+4x-4) + B(3x^2-2x) + C(x^2+2x)$

$x^2+4 = x^2(3A+3B+C) + x(4A-2B+2C) - 4A$

$1 = 3A + 3B + C \qquad 0 = 4A - 2B + 2C \qquad 4 = -4A$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad A = -1$

$1 = -3 + 3B + C \qquad 0 = -4 - 2B + 2C \qquad B = \frac{1}{2}$

$\quad 4 = 3B + C \qquad\quad 4 = -2B + 2C \qquad\quad C = \frac{5}{2}$

(Do substitution or elimination)

Ex #3 $\dfrac{x^2+4x+1}{x^3-x^2+x-1} = \dfrac{x^2+4x+1}{x^2(x-1)+1(x-1)} = \dfrac{x^2+4x+1}{(x-1)(x^2+1)}$

$\dfrac{x^2+4x+1}{(x-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1}$

$x^2+4x+1 = Ax^2+A + Bx^2-Bx+Cx-C$

$x^2+4x+1 = x^2(A+B)+x(C-B)+A-C$

$1 = A+B \qquad 4 = C-B \qquad\qquad 1 = A-C$

$\qquad\qquad\qquad B = C-4 \qquad\qquad A = 1+C$

$1 = 1+C+C-4$

$4 = 2C$

$C = 2 \qquad\qquad B = -2 \qquad\qquad A = 3$

$$\boxed{\dfrac{x^2+4x+1}{x^3-x^2+x-1} = \dfrac{3}{x-1} + \dfrac{-2x+2}{x^2+1}}$$

Ex #4 $\dfrac{2x^3-x^2+5x}{(x^2+1)^2} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x^2+1)^2}$

$2x^3-x^2+5x = Ax(x^4+2x^2+1)+B(x^4+2x^2+1)+Cx^3+Cx$

$\qquad\qquad\qquad\qquad\qquad\qquad +Dx^2+D$

oops! $\quad 2x^3-x^2+5x = Ax^5+2Ax^3+Ax+Bx^4+2Bx^2+B+Cx^3+Cx$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad + Dx^2+D$

$2x^3-x^2+5x = Ax^5+Bx^4+x^3(2A+C)+$

$A = 2 \qquad B = -1 \qquad C = 3 \qquad D = 1$

$$\boxed{\dfrac{2x^3-x^2+5x}{(x^2+1)^2} = \dfrac{2x-1}{x^2+1} + \dfrac{3x+1}{(x^2+1)^2}}$$