5.4 Multiple-Angle Identities

Double-Angle Identities

$$\sin 2u = 2\sin u\cos u$$

$$\cos 2u = \begin{cases} \cos^2 u - \sin^2 u \\ 2\cos^2 u - 1 \\ 1 - 2\sin^2 u \end{cases}$$

$$\tan 2u = \frac{2\tan u}{1-\tan^2 u}$$

Half-Angle Identities

$$\sin \frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$$

$$\cos \frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan \frac{u}{2} = \begin{cases} \pm\sqrt{\dfrac{1-\cos u}{1+\cos u}} \\ \dfrac{1-\cos u}{\sin u} \\ \dfrac{\sin u}{1+\cos u} \end{cases}$$

Power-Reducing Identities

$$\sin^2 u = \frac{1-\cos 2u}{2}$$

$$\cos^2 u = \frac{1+\cos 2u}{2}$$

$$\tan^2 u = \frac{1-\cos 2u}{1+\cos 2u}$$

Ex #1 Prove $\cos^4 x - \sin^4 x = \cos 2x$.

$$\cos^4 x - \sin^4 x = \cos 2x$$
$$\cos^4 x - \sin^4 x = (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)$$
$$= (1)(\cos^2 x - \sin^2 x)$$
$$\cos^4 x - \sin^4 x = \cos 2x \quad \blacksquare$$

Ex #2 Rewrite $\cos^4 x$ in terms of trig functions with no power greater than 1.

$$\cos^4 x = (\cos^2 x)^2$$
$$= \left(\frac{1+\cos 2x}{2}\right)^2 \qquad (1+\cos 2x)(1+\cos 2x)$$
$$= \frac{1 + 2\cos 2x + \cos^2 2x}{4}$$
$$= \frac{1}{4}\left(1 + 2\cos 2x + \cos^2 2x\right)$$
$$= \frac{1}{4}\left(1 + 2\cos 2x + \frac{1+\cos 4x}{2}\right)$$
$$\boxed{\cos^4 x = \frac{1}{4} + \frac{\cos 2x}{2} + \frac{1+\cos 4x}{8}}$$

Ex #3 Show $\sin^2\left(\frac{\pi}{8}\right) = \frac{2-\sqrt{2}}{4}$.

$$\sin^2\left(\frac{\pi}{8}\right) = \frac{1-\cos 2\left(\frac{\pi}{8}\right)}{2}$$
$$= \frac{1-\cos\frac{\pi}{4}}{2}$$
$$= \frac{1}{2}\left(1-\cos\frac{\pi}{4}\right)$$
$$= \frac{1}{2}\left(1-\frac{\sqrt{2}}{2}\right)$$
$$= \frac{1}{2} - \frac{\sqrt{2}}{4}$$
$$= \frac{2}{4} - \frac{\sqrt{2}}{4}$$
$$\boxed{\sin^2\left(\frac{\pi}{8}\right) = \frac{2-\sqrt{2}}{4}} \quad \blacksquare$$

Ex #4 Solve $\sin 2x = \cos x$ on $[0, 2\pi)$.

$$\sin 2x = \cos x$$

$$2\sin\cancel{x}\cos\cancel{x} = \cos x$$

$$2\sin x\cos x - \cos x = 0$$

$$\cos x(2\sin x - 1) = 0$$

$$\cos x = 0 \qquad 2\sin x - 1 = 0$$

$$x = \cos^{-1} 0 \qquad 2\sin x = 1$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad \sin x = \frac{1}{2}$$

$$x = \sin^{-1}\left(\frac{1}{2}\right)$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\boxed{x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}}$$

Ex #5 Solve $\sin^2 x = 2\sin^2 \frac{x}{2}$.

$$\sin^2 x = 2\sin^2\left(\frac{x}{2}\right)$$

$$\sin^2 x = \cancel{2}\left(\frac{1 - \cos 2\left(\frac{x}{2}\right)}{\cancel{2}}\right)$$

$$\sin^2 x = 1 - \cos x$$

$$1 - \cos^2 x = 1 - \cos x$$

$$0 = \cos^2 x - \cos x$$

$$0 = \cos x(\cos x - 1)$$

$$0 = \cos x \qquad 0 = \cos x - 1$$

$$x = \cos^{-1} 0 \qquad \cos x = 1$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad x = \cos^{-1} 1$$

$x = 0$ ~~2π~~ ← don't need 2π.

$$\boxed{\begin{array}{l} x = \frac{\pi}{2} + 2\pi n, n \in \mathbb{Z} \\ x = \frac{3\pi}{2} + 2\pi n, n \in \mathbb{Z} \quad x = 2\pi n, n \in \mathbb{Z} \end{array}}$$