10.3 More on Limits

Calculus

Derivatives $\rightarrow$ slope of the tangent line

Integrals $\rightarrow$ area under the curve

Derivatives are introduced as limits.

Notation

$\lim_{x \to \infty} f(x) = L$ "The limit of f(x) as x approaches infinity is L".

$\lim_{x \to a^-} f(x) = L$ "The limit of f(x) as x approaches a, on the left, is L"

$\lim_{x \to a^+} f(x) = L$ "The limit of f(x) a x approaches a, on the right, is L."

Limit Existance

$$L = \lim_{x \to a} f(x) \overset{\text{iff}}{\longleftrightarrow} \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

Algebraically ☆best way☆

Direct Substitution

$$\lim_{x \to -1} 2x^2 = 2(-1)^2$$

$$\boxed{\lim_{x \to -1} 2x^2 = 2}$$

Algebraic

difference of cubes

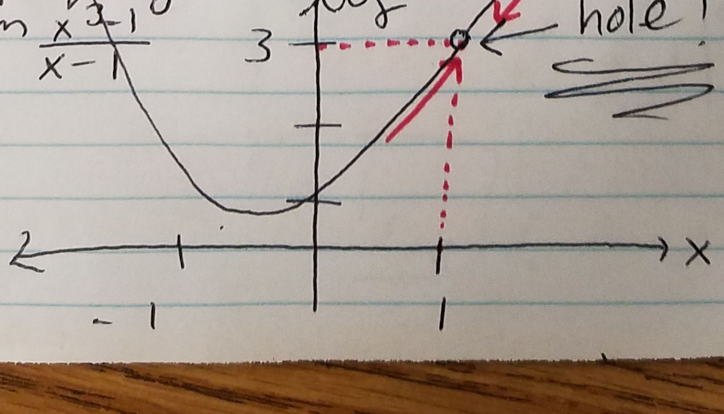
$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x^2 + x + 1)}{\cancel{x-1}}$$

$$= \lim_{x \to 1} (x^2 + x + 1)$$

$$\boxed{\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3}$$

Graphically

$\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$



$$\boxed{\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3}$$

Numerically

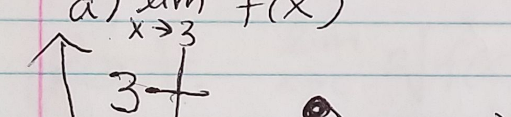
$\lim_{x\to 1} \frac{x^3-1}{x-1} = \frac{?}{0}$

x	$Y = \frac{x^3-1}{x-1}$
.997	2.991
.998	2.994
.999	2.997
1	ERROR = undefined (hole)
1.001	3.003
1.002	3.006
1.003	3.009

$\boxed{\lim_{x\to 1} \frac{x^3-1}{x-1} = 3}$

Ex # 1

a) $\lim_{x\to 3} f(x)$

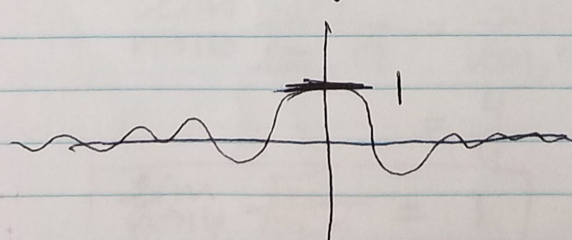


$\boxed{\lim_{x\to 3} f(x)}$ DNE

b) $\boxed{\lim_{x\to 1} (x^2+3x-4) = 0}$

c) $\lim_{x\to 0} \frac{\sin x}{x}$ = Indeterminant

(make a graph or a table)



X	y
-.002	1
-.001	1
0	error
.001	1
.002	1

$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$

Properties of Limits

Sum. $\lim_{x\to c}\left(f(x)+g(x)\right)=\lim_{x\to c}f(x)+\lim_{x\to c}g(x)$

Diff. $\lim_{x\to c}\left(f(x)-g(x)\right)=\lim_{x\to c}f(x)-\lim_{x\to c}g(x)$

Const. $\lim_{x\to c}\left(k\,g(x)\right)=K\lim_{x\to c}g(x)$

Prod. $\lim_{x\to c}\left(f(x)\cdot g(x)\right)=\lim_{x\to c}f(x)\cdot\lim_{x\to c}g(x)$

Quot. $\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{\lim_{x\to c}f(x)}{\lim_{x\to c}g(x)}$

Pow. $\lim_{x\to c}(f(x))^n=\left(\lim_{x\to c}f(x)\right)^n$

Root. $\lim_{x\to c}\sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to c}f(x)}$

Ex #2 Find $\lim_{x\to 0}\frac{e^x-\tan x}{\cos^2 x}$.

$$\lim_{x\to 0}\frac{e^x-\tan x}{\cos^2 x}=\frac{e^0-\tan 0}{\cos^2 0}$$

$$=\frac{1-0}{1^2}$$

$$\boxed{\lim_{x\to 0}\frac{e^x-\tan x}{\cos^2 x}=1}$$

Ex #3 Find $\lim_{n\to 16}\frac{\sqrt{n}}{\log_2 n}$.

$$\lim_{n\to 16}\frac{\sqrt{n}}{\log_2 n}=\frac{\sqrt{16}}{\log_2 16}$$

$$=\frac{4}{4}$$

$$\boxed{\lim_{n\to 16}\frac{\sqrt{n}}{\log_2 n}=1}$$

Ex #4 Find $\lim_{x\to 3}f(x)$ & prove it is discon't @ $x=3$.

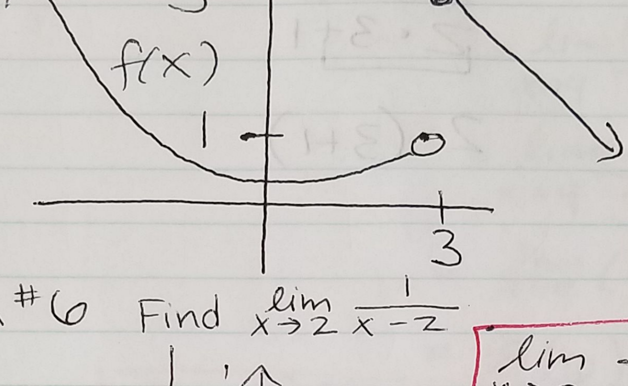
$$f(x)=\begin{cases}2, & x=3\\ \frac{x^2-9}{x-3}, & x\neq 3\end{cases}$$

$$\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)}{x-3}$$

$$=\lim_{x\to 3}(x+3)$$

$$\boxed{\lim_{x\to 3}\frac{x^2-9}{x-3}=6\neq 2=f(3)}$$ $\therefore$ it is discon't

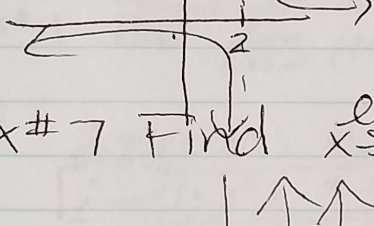
Ex #5



Find $\lim_{x \to 3^-} f(x)$, $\lim_{x \to 3^+} f(x)$, $\lim_{x \to 3} f(x)$.

$$\boxed{\begin{array}{l} \lim_{x \to 3^-} f(x) = 1 \\ \lim_{x \to 3^+} f(x) = 3 \\ \lim_{x \to 3} f(x) \text{ DNE} \end{array}}$$

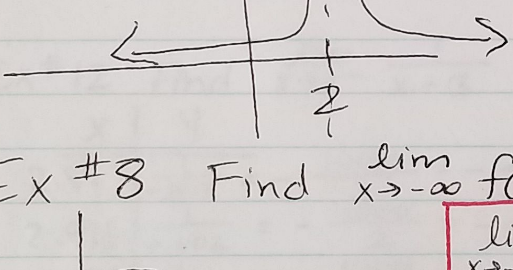
Ex #6 Find $\lim_{x \to 2} \frac{1}{x-2}$



$$\boxed{\lim_{x \to 2} \frac{1}{x-2} \text{ DNE}}$$

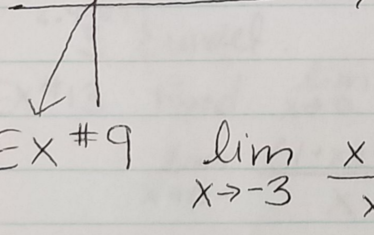
because $\lim_{x \to 2^-} \frac{1}{x-2} \neq \lim_{x \to 2^+} \frac{1}{x-2}$

Ex #7 Find $\lim_{x \to 2} \frac{1}{(x-2)^2}$.



$$\boxed{\lim_{x \to 2} \frac{1}{(x-2)^2} = \infty}$$

Ex #8 Find $\lim_{x \to -\infty} f(x)$ & $\lim_{x \to \infty} f(x)$ if $f(x) = xe^{-x}$.



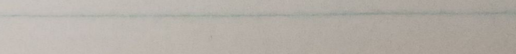
$$\boxed{\begin{array}{l} \lim_{x \to -\infty} f(x) = -\infty \\ \lim_{x \to \infty} f(x) = 0 \end{array}}$$

Ex #9 $\lim_{x \to -3} \frac{x^2+7x+12}{x^2-9} = \lim_{x \to -3} \frac{(x+3)(x+4)}{(x+3)(x-3)}$

$= \lim_{x \to -3} \frac{x+4}{x-3}$

$$\boxed{\lim_{x \to -3} \frac{x^2+7x+12}{x^2-9} = -\frac{1}{6}}$$

Ex #10



Ex#10 $\lim\limits_{x\to 9} \dfrac{x^2-81}{\sqrt{x}-3} = \lim\limits_{x\to 9} \dfrac{x^2-81}{\sqrt{x}-3}\left(\dfrac{\sqrt{x}+3}{\sqrt{x}+3}\right)$

$= \lim\limits_{x\to 9} \dfrac{(x^2-81)(\sqrt{x}+3)}{x \cancel{+3\sqrt{x}-3\sqrt{x}} - 9}$

$= \lim\limits_{x\to 9} \dfrac{(x+9)\cancel{(x-9)}(\sqrt{x}+3)}{\cancel{x-9}}$

$= \lim\limits_{x\to 9} (x+9)(\sqrt{x}+3)$

$= 18(6)$

$$\boxed{\lim_{x\to 9} \frac{x^2-81}{\sqrt{x}-3} = 108}$$

Ex#11 Find $\lim\limits_{x\to\pm\infty} (1+3^x)$ algebraically.

$\lim\limits_{x\to\infty} (1+3^x) = 1+3^{\infty}$

$$\boxed{\lim_{x\to\infty} (1+3^x) = \infty}$$

$\lim\limits_{x\to -\infty} (1+3^x) = 1+3^{-\infty}$

$= 1 + \dfrac{1}{3^{\infty}}$

$= 1+0$

$$\boxed{\lim_{x\to -\infty} (1+3^x) = 1}$$

Ex#12 Find $\lim\limits_{x\to 3^-} \dfrac{1}{x-3}$ numerically.

x	y
2.997	$\frac{1}{-.003} = -\frac{1000}{3} = -\cancel{333.}\overline{33}$
2.998	$\frac{1}{-.002} = -\frac{1000}{2} = -500$
2.999	$\frac{1}{-.001} = -\frac{1000}{1} = -1000$
2.9995	
3	undef.

$$\boxed{\lim_{x\to 3^-} \frac{1}{x-3} = -\infty}$$

Ex#13 Find $\lim\limits_{x\to 0} \dfrac{(1+x)^3-1}{x}$ algebraically.

$\lim\limits_{x\to 0} \dfrac{(1+x)^3-1}{x} = \lim\limits_{x\to 0} \dfrac{(1+x)(1+x)(1+x)-1}{x}$

$= \lim\limits_{x\to 0} \dfrac{\cancel{1}+3x+3x^2+x^3\cancel{-1}}{x}$

$= \lim\limits_{x\to 0} (3+3x+x^2)$

$$\boxed{\lim_{x\to 0} \frac{(1+x)^3-1}{x} = 3}$$

Ex #14 Find $\lim_{x\to 0} \frac{|x|}{x^2}$ algebraically.

$$\lim_{x\to 0} \frac{|x|}{x^2} = \lim_{x\to 0} \begin{cases} \frac{x}{x^2}, & x \geq 0 \\ \frac{-x}{x^2}, & x < 0 \end{cases}$$

$$= \begin{cases} \lim_{x\to 0^+} \left(\frac{1}{x}\right) = \infty \\ \lim_{x\to 0^-} \left(-\frac{1}{x}\right) = \infty \end{cases}$$

$$\boxed{\lim_{x\to 0} \frac{|x|}{x^2} = \infty}$$

Ex #15 Determine $\lim_{x\to\infty} \frac{3x^4-2x^3+x^2+5}{5x^4-9}$ algebraically.

$$\lim_{x\to\infty} \frac{3x^4-2x^3+x^2+5}{5x^4-9} = \lim_{x\to\infty} \frac{3-2\frac{1}{x}+\frac{1}{x^2}+\frac{5}{x^4}}{5-\frac{9}{x^4}}$$

$$= \frac{3-0+0+0}{5-0}$$

$$\boxed{\lim_{x\to\infty} \frac{3x^4-2x^3+x^2+5}{5x^4-9} = \frac{3}{5}}$$