2.10 | 2.11

A normal line is perpendicular to the tangent line.
Use the opposite-reciprocal of the tangent slope.

Implicit Differentiation

Use this to take derivatives of relations.
Must take the derivative WRT x (you will end up with some $\frac{dy}{dx}$ or y' terms).
Then isolate the $\frac{dy}{dx} = y'$ term & you're done.

Ex#1 Find y' for $x + \sin y = xy$.

$$\frac{d}{dx}(x+\sin y) = \frac{d}{dx}(xy)$$
$$1 + \cos y \cdot y' = y + xy'$$
$$\cos y \cdot y' - xy' = y - 1$$
$$y'(\cos y - x) = y - 1$$
$$\boxed{y' = \frac{y-1}{\cos y - x}}$$

Ex#2 Find $\frac{d^2y}{dx^2}$ for $2x^3 - 3y^2 = 8$.

$$6x^2 - 6y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{x^2}{y}$$
$$\frac{d^2y}{dx^2} = \frac{2xy - x^2\frac{dy}{dx}}{y^2}$$
$$\frac{d^2y}{dx^2} = \left(\frac{2xy - x^2\left(\frac{x^2}{y}\right)}{y^2}\right)\left(\frac{y}{y}\right)$$
$$\boxed{\frac{d^2y}{dx^2} = \frac{2xy^2 - x^4}{y^3}}$$

Ex#3 Find $\frac{dy}{dx}$ for $x^2 - xy + y^2 = 3$.

$$\boxed{\frac{dy}{dx} = \frac{y-2x}{2y-x}}$$

$$2x - \left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$$

$$2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{y-2x}{2y-x}$$

$$\left.\frac{dy}{dx}\right|_{(5,2)} = \frac{2-2(5)}{2(2)-5}$$

Ex#4 Find where $x^3 + y^3 - 9xy = 0$ has a horizontal tangent.

$$3x^2 + 3y^2\frac{dy}{dx} - 9\left(y + x\frac{dy}{dx}\right) = 0$$

$$3x^2 + 3y^2\frac{dy}{dx} - 9y - 9x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{9y-3x^2}{3y^2-9x}$$

$$0 = 9y - 3x^2$$

$$3x^2 = 9y$$

$$y = \frac{x^2}{3}$$

$$x^3 + \left(\frac{x^2}{3}\right)^3 - 9x\left(\frac{x^2}{3}\right) = 0$$

$$x^3 + \frac{x^6}{27} - 3x^3 = 0$$

$$\frac{x^6}{27} - 2x^3 = 0$$

$$x^3\left(\frac{x^3}{27} - 2\right) = 0$$

$$x^3 = 0 \quad \text{or} \quad \frac{x^3}{27} - 2 = 0$$

$$x^3 = 54$$

$$\boxed{x \neq 0 \quad \text{or} \quad x = 3\sqrt[3]{2}}$$