6.2 Dot Product (Actual Vector Multiplication)

Dot Product AKA Inner Product

$\vec{u}\cdot\vec{v} = \langle u_1, u_2\rangle \cdot \langle v_1, v_2\rangle = u_1 v_1 + u_2 v_2$

Ex #1 $\langle 3,4\rangle \cdot \langle 5,2\rangle = 23$

$\langle 1,-2\rangle \cdot \langle -4,3\rangle = -10$

$(2i-j)\cdot(3i-5j) = 11$

Properties of the Dot Product

$\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ $\quad$ $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$

$\vec{u}\cdot\vec{u} = |\vec{u}|^2$ $\quad$ $(\vec{u}+\vec{v})\vec{w} = \vec{u}\vec{w} + \vec{v}\cdot\vec{w}$

$0\cdot\vec{u} = \vec{0}$ $\quad$ $(c\vec{u})\cdot\vec{v} = \vec{u}(c\vec{v}) = c(\vec{u}\cdot\vec{v})$

Angles Between Vectors

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$$

Also $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$

Ex #2 Find the angle between $\vec{u} = \langle 2,3\rangle$ & $\vec{v} = \langle -2,5\rangle$.

$$\theta = \cos^{-1}\left(\frac{11}{\sqrt{13}\sqrt{29}}\right) \approx 55.491^\circ$$

Ex #3 Find the angle between $\vec{u} = \langle 2,1\rangle$ & $\vec{v} = \langle -1,-3\rangle$.

$$\theta = \cos^{-1}\left(\frac{-5}{\sqrt{5}\sqrt{10}}\right) \approx 135^\circ$$

Orthogonal (Perpendicular) Vectors

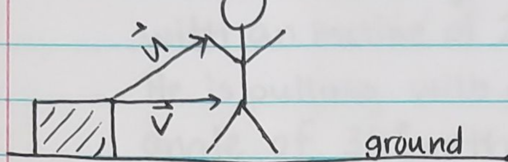
Perpendicular vectors have an angle of 90°, this means $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos 90^\circ = 0$.

Ex #4 Prove $\vec{u} = \langle 2,3\rangle$ & $\vec{v} = \langle -6,4\rangle$ are orthoganal.

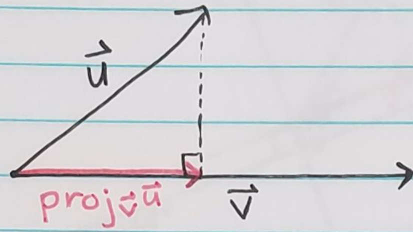
$\vec{u}\cdot\vec{v} = 2(-6) + 3(4) = -12 + 12 = 0$ ✓

Projecting one Vector Onto Another

"How much of one vector goes in the direction of the other vector?"



If I am pulling a box with force $\vec{u}$, I will have an effective force in the direction of $\vec{v}$. This is denoted $proj_{\vec{v}}\vec{u}$ (the projection of $\vec{u}$ onto $\vec{v}$.

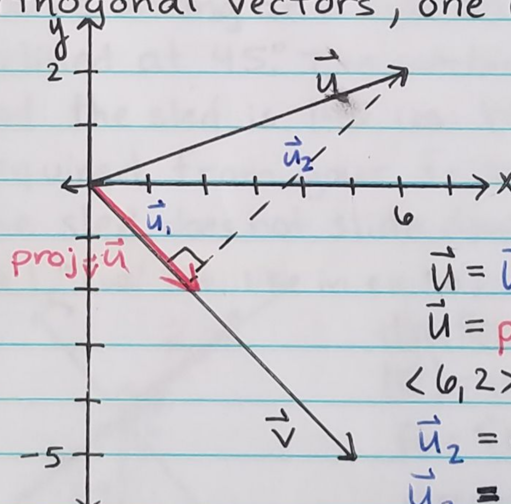


$$proj_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}$$

Ex #5 Find the projection of $\vec{u} = \langle 6,2\rangle$ onto $\vec{v} = \langle 5,-5\rangle$.

$proj_{\vec{v}}\vec{u} = \left(\frac{20}{50}\right)\langle 5,-5\rangle = \langle 2,-2\rangle$

Ex #6 Write $\vec{u}$ (from ex #5) as the sum of two orthogonal vectors, one which is $proj_{\vec{v}}\vec{u}$.



$\vec{u} = \vec{u}_1 + \vec{u}_2$

$\vec{u} = proj_{\vec{v}}\vec{u} + \vec{u}_2$

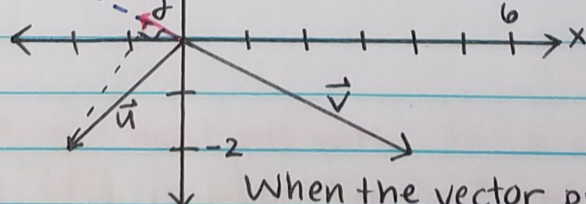
$\langle 6,2\rangle = \langle 2,-2\rangle + \vec{u}_2$

$\vec{u}_2 = \langle 6,2\rangle - \langle 2,-2\rangle$

$\vec{u}_2 = \langle 4,4\rangle$

$\vec{u} = \langle 2,-2\rangle + \langle 4,4\rangle$

Ex #7 Sketch $\vec{u} = \langle -2,-2\rangle$ & $\vec{v} = \langle 4,-2\rangle$. Find and sketch $proj_{\vec{v}}\vec{u}$ & explain why it is negative.



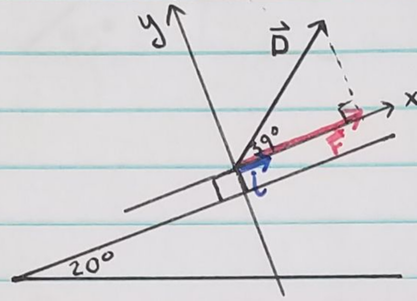
$proj_{\vec{v}}\vec{u} = \frac{-8+4}{16+4}\langle 4,-2\rangle$

$= -\frac{1}{5}\langle 4,-2\rangle$

$= \langle -\frac{4}{5}, \frac{2}{5}\rangle$

When the vector projection occurs, it goes in the opposite direction of $\vec{v}$. This creates a "opposite/negative" projection.

Ex #8 A father is pulling his daughter up a snowy hill with an incline of 20° while she is on a sled. He is pulling with a force of 100 lbs at an angle of 39° with respect to the hill. What is the effective force being exerted?



direction angle $\vec{D} = 39°$

$|\vec{D}| = 100$ lbs

$\vec{D} = \langle 100\cos 39°, 100\sin 39° \rangle$

direction angle $i = 0°$

$|i| = 1$

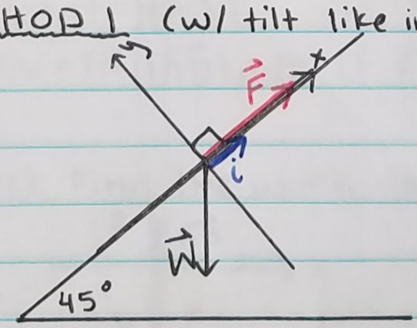
$i = \langle 1, 0 \rangle$

$|\vec{F}| = |\text{proj}_i \vec{D}| = 77.715$ lbs

Note: $|\vec{F}| = \text{proj}_i \vec{D}$ is also the "a" component of $\vec{D}$ since it is the horizontal portion of $\vec{D}$.

Ex #9 You're sitting on a sled on the side of a hill inclined at 45°. The combined weight of you and the sled is 140 lbs. What force is required from your friend to ensure the sled does not slide down the hill?

METHOD 1 (w/ tilt like in ex #8)



direction angle $\vec{W} = 225°$

$|\vec{W}| = 140$ lbs

$\vec{W} = \langle 140\cos 225°, 140\sin 225° \rangle$

direction angle $i = 0°$

$|i| = 1$

$i = \langle 1, 0 \rangle$

$\vec{F} = \text{proj}_i \vec{W} = \frac{-98.995}{1} \langle 1, 0 \rangle$

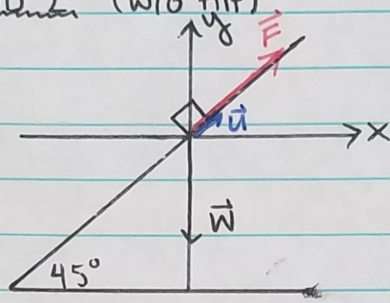
$\vec{F} = \langle -98.995, 0 \rangle$

$|\vec{F}| = 98.995$ lbs

Friend must pull with a force of 98.995 lbs.

Note: $|\vec{F}|$ is also the "a" component (absolute value) of $\vec{W}$.

METHOD 2 (w/o tilt)



direction angle $\vec{W} = 270°$

$|\vec{W}| = 140$ lbs

$\vec{W} = \langle 0, -140 \rangle$

direction angle $\vec{u} = 45°$

$|\vec{u}| = 1$

$\vec{u} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

$\vec{F} = proj_{\vec{u}} \vec{W} = \frac{-140 \frac{\sqrt{2}}{2}}{1} \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

$= \langle -70, -70 \rangle$

$|\vec{F}| = 98.995$ lbs

Friend must pull with a force of 98.995 lbs.

Note: Using the "a" component of $\vec{W}$ doesn't work here because $\vec{u}$ is not the horizontal portion of $\vec{W}$.

Work

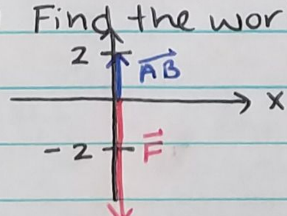
Work is the force required to move an object over a certain distance & direction.

$W = \vec{F} \cdot \vec{AB}$ where $\vec{F}$ is force vector & $\vec{AB}$ is the displacement vector

$W = |\vec{F}||\vec{AB}|$ if $\vec{F}$ is in the same direction as $\vec{AB}$

$W = |\vec{F}||\vec{AB}| \cos\theta$ if $\vec{F}$ & $\vec{AB}$ have θ between them

Ex# 10 Find the work done lifting a 15 lbs backpack 2 ft.

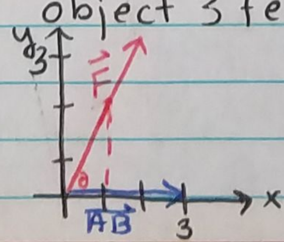


$w = |F||AB|$

$= 15_{lbs}(2_{ft})$

$= 30$ foot-pounds

Ex# 11 Find the work done by a 10 lbs force acting in the direction $\langle 1, 2 \rangle$ in moving an object 3 feet from (0,0) to (3,0).



$\vec{AB} = \langle 3, 0 \rangle$ $\theta = \tan^{-1}(2)$

$\vec{F} = \langle 10 \cos(\tan^{-1}(2)), 10 \sin(\tan^{-1}(2)) \rangle$

$W = \vec{F} \cdot \vec{AB} = 3(10)\cos(\tan^{-1}(2)) + 0$

$W = 13.416$ foot-pounds