7.4

Special Series

Geometric: $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^{n} = \frac{a}{1-r}$ if $|r|<1$

otherwise the series diverges.

Harmonic: $\sum_{n=1}^{\infty} \frac{1}{n}$ always diverges

★ $\sum_{n=1}^{\infty} \frac{1}{n^2}$ always converges ★

Telescoping: when every term in a series except for a few terms cancel out.

Ex#1 $\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2} = \sum_{n=0}^{\infty} \left[\frac{1}{(n+2)(n+1)}\right]$

$$= \sum_{n=0}^{\infty} \left[\frac{-1}{n+2} + \frac{1}{n+1}\right]$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{n+1} - \frac{1}{n+2}\right]$$

$$= \left(1-\cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}}-\cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}}-\cancel{\tfrac{1}{4}}\right) + \left(\cancel{\tfrac{1}{4}}-\cancel{\tfrac{1}{5}}\right) +$$

$$+ \cdots + \left(\cancel{\tfrac{1}{n}} - \tfrac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} = 1 - 0 = \boxed{1}$$

$1 = An + A + Bn + 2B$

$0 = A + B \qquad 1 = A + 2B$

$A = -B \qquad 1 = B$

$-1 = A$

Integral Test

Suppose $f(x)$ is continuous, positive, & decreasing on $[k,\infty)$ & that $f(n)=a_n$

① $\int_k^{\infty} f(x)\,dx$ is convergent, then so is $\sum_{n=k}^{\infty} a_n$.

② $\int_k^{\infty} f(x)\,dx$ is divergent, then so is $\sum_{n=k}^{\infty} a_n$.

Ex#2 Determine if $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ converges.

$$\int_2^{\infty} \frac{dx}{x\ln x} = \lim_{b\to\infty} \int_2^{b} \frac{dx}{x\ln x}$$

$$= \lim_{b\to\infty} \left[\ln(\ln x)\right]_2^{b}$$

$$= \lim_{b\to\infty} \left(\ln(\ln b) - \ln(\ln 2)\right)$$

$$= \infty$$

so by the integral test, $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ diverges.

p-Series Test

If $k>0$, then $\sum_{n=k}^{\infty} \frac{1}{n^p}$ converges if $p>1$ & div. if $p \le 1$.

Ex#3 $\sum_{n=4}^{\infty} \frac{1}{n^7}$ converges by the p-series test.

Divergence Test

$\lim_{n\to\infty} a_n \neq 0$ then the series diverges.

Direct Comparison Test

If a_n & b_n are positive & $a_n \leq b_n$, then if

1. $\sum_{n=0}^{\infty} b_n$ is convergent, then $\sum a_n$ must converge.
2. $\sum a_n$ is divergent, then $\sum b_n$ must diverge.

Ex #4 $\sum_{n=1}^{\infty} \frac{n}{n^2-\cos^2 n}$ compare it to $\frac{n}{n^2}$

$$\frac{n}{n^2-\cos^2 n} > \frac{n}{n^2} = \frac{1}{n}$$ prove it will diverge

$\sum_{n=1}^{\infty} \frac{n}{n^2-\cos^2 n}$ is divergent by the direct comparison test & $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent because it's the harmonic series.

Limit Comparison Test

Assuming a_n & b_n are nonnegative, let $c = \lim_{n\to\infty} \frac{a_n}{b_n}$. If c is positive & finite,

then both a_n & b_n converge or both a_n & b_n diverge.

Ex # 5 $\sum_{n=0}^{\infty} \frac{1}{3^n - n}$ (similar to $\frac{1}{3^n}$)

$$c = \lim_{n\to\infty} \frac{3^n}{3^n - n} = \lim_{n\to\infty} \frac{1}{1-\frac{n}{3^n}} = \frac{1}{1-\lim_{n\to\infty}\frac{n}{3^n}}$$

$$= \frac{1}{1-\lim_{n\to\infty}\frac{1}{(\ln 3)3^n}} = \frac{1}{1-0} = 1$$

Either both a_n & b_n converge or both " " diverge.

$\sum_{n=0}^{\infty} \frac{1}{3^n}$ is geometric w/ $|r|<1$ $\therefore$ converges & $\therefore$ $\sum_{n=0}^{\infty} \frac{1}{3^n-n}$ converges by Limit comparison test.

Ex #6 What function would you select for $\sum_{n=2}^{\infty} \frac{4n^2+n}{\sqrt[3]{n^7+n^3}}$ for LCT?

compare to $\frac{n^2}{\sqrt[3]{n^7}} = \frac{n^2}{n^{7/3}} = \frac{1}{n^{1/3}}$

Ratio Test

Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$
- $L > 1$ diverges
- $L < 1$ converges
- $L = 1$ use another test

Ex # 7 $\sum_{n=0}^{\infty} \frac{n!}{5^n}$

$$\lim_{n\to\infty}\left|\frac{(n+1)!}{5^{n+1}}\left(\frac{5^n}{n!}\right)\right| = \lim_{n\to\infty}\left|\frac{(n+1)\cancel{n(n-1)\cdots}}{5\cdot\cancel{5^n}}\left(\frac{\cancel{5^n}}{\cancel{n(n-1)(n-2)}}\right)\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{5}\right|$$

$$= \infty > 1$$

By Ratio test $\sum_{n=0}^{\infty}\frac{n!}{5^n}$ diverges.

Root Test

Let $L = \lim_{n\to\infty}\sqrt[n]{|a_n|}$
- $L > 1$ diverges
- $L < 1$ converges
- $L = 1$ use another test

Ex # 8 $\sum_{n=0}^{\infty}\left(\frac{5n-3n^3}{7n^3+2}\right)^n$

$$\lim_{n\to\infty}\sqrt[\cancel{n}]{\left|\left(\frac{5n-3n^3}{7n^3+2}\right)^{\cancel{n}}\right|} = \lim_{n\to\infty}\left|\frac{5n-3n^3}{7n^3+2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\frac{5}{n^2}-3}{7+\frac{2}{n^3}}\right|$$

$$= \left|\frac{-3}{7}\right|$$

$$= 3/7 < 1$$

By Root test $\sum_{n=0}^{\infty}\left(\frac{5n-3n^3}{7n^3+2}\right)^n$ converges.

Guideline / Strategy

1. Divergence Test (only if very quick)
2. p-series / geometric
3. does it look like p-series / geometric? Direct Comparison
4. rational expression w/ polynomials / sq roots? direct / limit comparison
5. factorials or constant power of n? ratio test
6. can you write as $a_n = (b_n)^n$? root test
7. Integral test