2.5 Complex Zeros & the Fundamental Theorem of Algebra

Fundamental Theorem of Algebra

A polynomial function of degree n has n complex zeros (real, nonreal, and including repeats).

Linear Factorization Theorem $(ax+b)$

The polynomial $f(x)$ has exactly n linear factors and

$$f(x) = a(x-z_1)(x-z_2)\cdots(x-z_n)$$

with $a \to$ leading coefficient &

$z_1, z_2, \ldots, z_n \to$ complex zeros.

* Note: The graph will not cross the x-axis for imaginary zeros.

Ex #1 Write in std form, ID zeros, state x-int.

$$f(x) = (x-3i)(x+3i)(x+1)(x-\sqrt{2})(x+\sqrt{2})$$

$$= (x^2 \cancel{+3xi} \cancel{-3xi} - 9i^2)(x+1)(x^2 \cancel{+\sqrt{2}x} \cancel{-\sqrt{2}x} - \sqrt{2}^2)$$

$$= (x^2 - 9(-1))(x+1)(x^2 - 2)$$

$$= (x^2+9)(x+1)(x^2-2)$$

$$= (x+1)(x^4 - 2x^2 + 9x^2 - 18)$$

$$= (x+1)(x^4 + 7x^2 - 18)$$

$$= x^5 + 7x^3 - 18x + x^4 + 7x^2 - 18$$

$$f(x) = x^5 + x^4 + 7x^3 + 7x^2 - 18x - 18$$

Zeros: $3i, -3i, -1, \sqrt{2}, -\sqrt{2}$

x-int: $-1, \sqrt{2}, -\sqrt{2}$

Complex Conjugate Theorem

If $a+bi$ is a zero, then $a-bi$ is a zero.

(If $\sqrt{a}$ also have $-\sqrt{a}$)

Ex #2 Find the zeros for $f(x)=2x^5-6x^4-10x^3+10x^2-12x+16$.

$\frac{p}{q} = \pm \frac{1,16,2,8,4}{1,2}$

$$
\begin{array}{r|rrrrrr}
1 & 2 & -6 & -10 & 10 & -12 & 16 \\
 & & 2 & -4 & -14 & -4 & -16 \\
\hline
4 & 2 & -4 & -14 & -4 & -16 & | \checkmark \\
 & & 8 & 16 & 8 & 16 & \\
\hline
-2 & 2 & 4 & 2 & 4 & | \checkmark & \\
 & & -4 & 0 & -4 & & \\
\hline
 & 2 & 0 & 2 & | \checkmark & &
\end{array}
$$

$* \; a^2+b^2 = (a+bi)(a-bi)$

$f(x) = (x-1)(x-4)(x+2)(2x^2+2)$

$f(x) = 2(x-1)(x-4)(x+2)(x^2+1)$

$f(x) = 2(x-1)(x-4)(x+2)(x+i)(x-i)$

$\boxed{\text{zeros: } 1, 4, -2, -i, i}$

$x^2+1=0$

$x^2=-1$

$x=\pm\sqrt{-1}$

$x=\pm i$

Ex #3 Given that $x=3-2i$ is a zero of $f(x)=x^4-6x^3+11x^2+12x-26$, find all zeros & write the linear factorization.

$$
\begin{array}{r|rrrrr}
3-2i & 1 & -6 & 11 & 12 & -26 \\
 & & 3-2i & -13 & -6+4i & 26 \\
\hline
3+2i & 1 & -3-2i & -2 & 6+4i & | \checkmark \\
 & & 3+2i & 0 & -6-4i & \\
\hline
 & 1 & 0 & -2 & | \checkmark &
\end{array}
$$

$(3-2i)(-3-2i)$

$-9 \cancel{-6i} \cancel{+6i} +4i^2$

$-9-4$

-13

$(3-2i)(6+4i)$

$18 \cancel{+12i} \cancel{-12i} -8i^2$

$18+8$

26

$$f(x) = [x-(3-2i)][x-(3+2i)](x^2-2)$$

$$\boxed{\begin{array}{c} f(x) = (x-3+2i)(x-3-2i)(x-\sqrt{2})(x+\sqrt{2}) \\ \text{zeros: } 3-2i,\ 3+2i,\ \sqrt{2},\ -\sqrt{2} \end{array}}$$

Ex #4 Given that -1 is a zero w/ mult. 2 & $-2-i$ is a zero w/ mult. 1, write the simplest polynomial function in std. form.

$$f(x) = (x+1)^2(x-(-2-i))(x-(-2+i))$$
$$= (x+1)^2(x+2+i)(x+2-i)$$
$$= (x+1)^2(x^2+2x-xi+2x+4-2i+xi+2i-$$
$$= (x+1)^2(x^2+4x+5)$$
$$= (x^2+2x+1)(x^2+4x+5)$$
$$= x^4+4x^3+5x^2+2x^3+8x^2+10x+x^2+4x+5$$
$$\boxed{f(x) = x^4+6x^3+14x^2+14x+5}$$

Factors with Real Coefficients

Factors that do not involve imaginary parts. These are made up of linear factors & irreducible quadratic factors.

Ex #5 Factor $f(x) = 3x^3-2x^2+x-2$ into linear & irreducible quadratic factors w/ real coeffs.

$P/q = \pm \frac{1,2}{1,3}$

1	3	-2	1	-2
		3	1	2
	3	1	2	0 ✓

$$\boxed{f(x) = (x-1)(3x^2+x+2)}$$