9.2 Binomial Theorem

Ex #1: Expand $(a+b)^n$ for $n=0,1,2,3,4,$ & 5. Can you predict what $(a+b)^6$ will look like? (without multiplying)

$$(a+b)^0 = 1$$
$$(a+b)^1 = 1a + 1b$$
$$(a+b)^2 = 1a^2 + 2ab + 1b^2$$
$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$
$$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$
$$(a+b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$$
$$(a+b)^6 = 1a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + 1b^6$$

<u>Binomial Coefficients</u>

$$C(n,r) = {}_nC_r = \binom{n}{r} \longrightarrow \text{"n choose r"}$$

gives the coefficient of the r^{th} term of the n^{th} binomial expansion

Ex #2: Use combinations to expand $(x+y)^9$.

$$(x+y)^9 = \binom{9}{0}x^9y^0 + \binom{9}{1}x^8y^1 + \binom{9}{2}x^7y^2 + \binom{9}{3}x^6y^3 + \binom{9}{4}x^5y^4 + \binom{9}{5}x^4y^5 + \dots + \binom{9}{9}x^0y^9$$

$$\boxed{(x+y)^9 = 1x^9 + 9x^8y + 36x^7y^2 + 84x^6y^3 + 126x^5y^4 + 126x^4y^5 + 84x^3y^6 + 36x^2y^7 + 9xy^8 + 1y^9}$$

<u>Pascal's Triangle</u>

Triangular array of binomial coefficients. Get the next row by adding the two terms above it w/ 1 along edge.

$n=0$	1
$n=1$	1 1
$n=2$	1 2 1
$n=3$	1 3 3 1
$n=4$	1 4 6 4 1
$n=5$	1 5 10 10 5 1
$n=6$	1 6 15 20 15 6 1
$n=7$	1 7 21 35 35 21 7 1
$n=8$	1 8 28 56 70 56 28 8 1
	⋮
n^{th}	$\binom{n}{0}\binom{n}{1}\binom{n}{2} \cdots\cdots \binom{n}{n}$

Ex #3: Determine the coeff. of x^{10} in the expansion of $(x+2)^{15}$.

$$\binom{15}{5}x^{10}(2)^5 = \boxed{96{,}096}\,x^{10}$$

Binomial Theorem

For any $n \in \mathbb{Z}^+$,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n}b^n,$$

where $\binom{n}{r} = n! / (r!\,(n-r)!)$.

Ex #4: Expand $(2x - y^2)^4$ using the Binomial Theorem.

$$\left(2x + (-y^2)\right)^4 = \binom{4}{0}(2x)^4 + \binom{4}{1}(2x)^3(-y^2) + \binom{4}{2}(2x)^2(-y^2)^2 + \binom{4}{3}(2x)(-y^2)^3 + \binom{4}{4}(-y^2)^4$$

$$= 1(16x^4) + 4(8x^3)(-y^2) + 6(4x^2)(y^4) + 4(2x)(-y^6) + 1(y^8)$$

$$\boxed{(2x - y^2)^4 = 16x^4 - 32x^3y^2 + 24x^2y^4 - 8xy^6 + y^8}$$

Factorial Identities

For $n \in \mathbb{Z}^+$, $n \geq 1$, $n! = n(n-1)!$.

For $n \in \mathbb{Z}$, $n \geq 0$, $(n+1)! = (n+1)\,n!$.

Ex #5: Prove $\binom{n+1}{2} - \binom{n}{2} = n$ for $n \geq 2$.

$$\binom{n+1}{2} - \binom{n}{2} = \frac{(n+1)!}{2!\,(n+1-2)!} - \frac{n!}{2!\,(n-2)!}$$

$$= \frac{(n+1)\,n!}{2!\,(n-1)!} - \frac{n!}{2!\,(n-2)!}$$

$$= \frac{(n+1)\,n\cancel{(n-1)!}}{2!\,\cancel{(n-1)!}} - \frac{n(n-1)\cancel{(n-2)!}}{2!\,\cancel{(n-2)!}}$$

$$= \frac{(n+1)\,n}{2} - \frac{n(n-1)}{2}$$

$$= \frac{n^2+n}{2} - \frac{n^2-n}{2}$$

$$= \frac{2n}{2}$$

$$\binom{n+1}{2} - \binom{n}{2} = n \quad \blacksquare$$