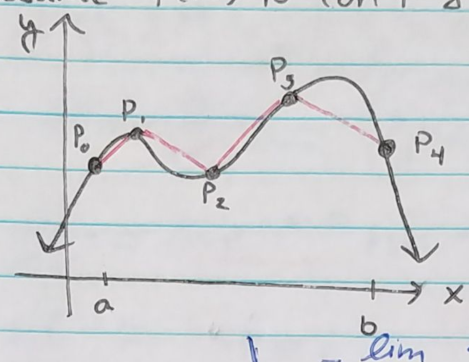
5.3/5.5

Arc Length

Assume $f(x)$ is con't & diff on $[a,b]$.



Estimating curve length by taking the sum of all the line segments $\overline{P_{i-1}P_i}$

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \overline{P_{i-1}P_i}$$

consider $\overline{P_{i-1}P_i} = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$

$= \sqrt{(\Delta x)^2 + (\underbrace{\Delta y}_{\text{varies!!}})^2}$

★ MVT states $f(b)-f(a) = m(b-a)$ ★

★ $f(x_i) - f(x_{i-1}) = f'(x^*)(x_i - x_{i-1})$ ★

★ $\Delta y_i = f'(x^*)\,\Delta x_i$ ★

$\overline{P_{i-1}P_i} = \sqrt{(\Delta x)^2 + [f'(x^*)\,\Delta x]^2}$

$= \sqrt{1 + [f'(x^*)]^2}\;\Delta x$

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x)]^2}\;\Delta x$$

$$\boxed{\begin{aligned} L &= \int_a^b \sqrt{1 + [f'(x)]^2}\,dx \\ L &= \int_c^d \sqrt{1 + [h'(y)]^2}\,dy \end{aligned}}$$

Ex#1 (1148) Find the length of the curve $x^2 + y^2 = 1$ in two ways, one involving an integral.

$y = \pm\sqrt{1-x^2}$ $\quad y' = \frac{1}{2}(1-x^2)^{-1/2}(2x) = \frac{-x}{\sqrt{1-x^2}}$

$L = 2\int_{-1}^{1} \sqrt{1 + \frac{x^2}{1-x^2}}\,dx$

$= 2\int_{-1}^{1} \sqrt{\frac{1}{1-x^2}}\,dx$

$= 2\sin^{-1}x \Big|_{-1}^{1}$

$= 2\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \boxed{2\pi}$

circle w/ r = 1 has C = 2π

Logistic Curve

$$\boxed{P(t) = \frac{m}{1 + Ce^{-rmt}}}$$

m = carrying capacity
C = constant determined by the initial condition
r = growth constant

$$P = \frac{m}{1+Ce^{-rmt}} \left(\frac{e^{rmt}}{e^{rmt}}\right) = \frac{me^{rmt}}{e^{rmt} + C} = Me^{rmt}(e^{rmt}+C)^{-1}$$

$$\frac{dP}{dt} = Me^{rmt}\, rm\, (e^{rmt}+C)^{-1} - (e^{rmt}+C)^{-2} e^{rmt}\, rmMe^{rmt}$$

$$\frac{dP}{dt} = Prm - P^2 r$$

$$\frac{dP}{dt} = Pr(m-P)$$

$$\boxed{\frac{dP}{dt} = r(m-P)P}$$

1. as $t \to \infty$, $P(t) \to m$
2. $\frac{dP}{dt}$ changes the fastest when $P = \frac{m}{2}$