Trigonometry

6

Unit Overview

In this unit you will build on your understanding of right triangle trigonometry as you study angles in radian measure, trigonometric functions, and periodic functions. You will investigate in depth the graphs of the sine, cosine, and tangent functions as well as trigonometric identities and reciprocal identities.

Key Terms

As you study this unit, add these and other terms to your math notebook. Include in your notes your prior knowledge of each word, as well as your experiences in using the word in different mathematical examples. If needed, ask for help in pronouncing new words and add information on pronunciation to your math notebook. It is important that you learn new terms and use them correctly in your class discussions and in your problem solutions.

Academic Vocabulary

- constraint

Math Terms

- arc length
- unit circle
- radian
- standard position
- initial side
- terminal side
- coterminal angles
- reference angle
- trigonometric function
- periodic function
- period
- amplitude
- midline
- phase shift

ESSENTIAL QUESTIONS

? What types of real-world problems can be modeled and solved using trigonometry?

? How are trigonometric functions used to model real-world problems?

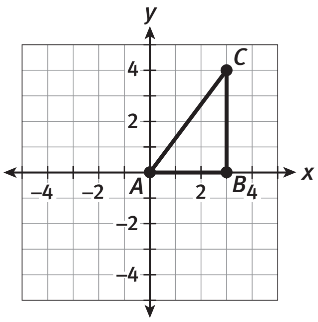
EMBEDDED ASSESSMENTS

This unit has two embedded assessments, following Activities 33 and 35. By completing these embedded assessments, you will demonstrate your understanding of trigonometric and circular functions.

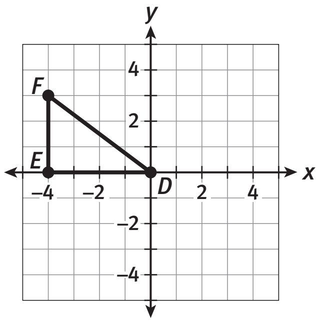
Getting Ready

Write your answers on notebook paper.
Show your work.

1. Find the length of the hypotenuse of a 30°-60°-90° triangle whose shorter leg is 3 units long.
2. Find the length of one of the legs of a 45°-45°-90° triangle whose hypotenuse is 6 units long.
3. Explain how the graph of $y = \frac{1}{4}(x + 1)^2 + 2$ differs from the graph of $y = x^2$. Explain how you can determine the differences without graphing.
4. Identify the coordinates of point C.



5. Identify the coordinates of point F.



6. Determine the circumference of a circle with a 7.4-centimeter radius. Use 3.14 for π. Round to the nearest hundredth.
7. Determine the circumference of a circle with a 2-inch diameter. Write your answer in terms of π.
8. Write a function $C(t)$ to represent the cost of a taxicab ride, where the charge includes a fee of \$2.75 plus \$0.45 for each tenth of a mile t. Then give the slope and y-intercept of the graph of the function.

Understanding Radian Measure

Revolving Restaurant
Lesson 31-1 Radian Measure

My Notes

Learning Targets:

- Develop formulas for the length of an arc.
- Describe radian measure.

SUGGESTED LEARNING STRATEGIES: Visualization, Predict and Confirm, Look for a Pattern, Create Representations, Sharing and Responding

An architecture firm is designing a circular restaurant that has a radius of 50 feet. It will be situated on top of a tall building, where it will rotate.

The lead architect wants to determine how far people seated at different distances from the center of the restaurant will travel as the restaurant rotates through various angles. To start, he will determine how far a customer seated at the window has traveled after a 60° rotation.

1. **Attend to precision.** How far from the center is a customer seated at the window? Find the circumference of a circle with this distance as the radius. Give an exact answer in terms of π.

MATH TIP

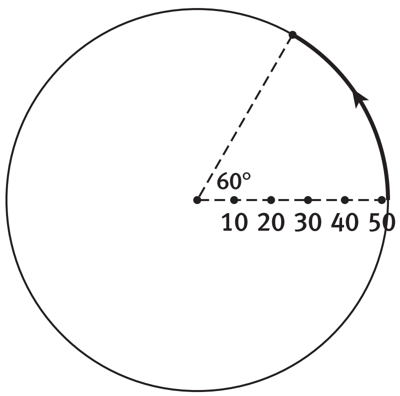
Use the formula $C = 2\pi r$ to find circumference.

2. What portion of the circumference of the circle is generated by a 60° rotation of the radius?

3. Use the portion of the circle generated by a 60° rotation of the restaurant to find the approximate distance traveled by this customer.

MATH TIP

π (*pi*) is an irrational number. If you need to provide the exact value of an expression that contains π, leave the symbol in the answer. We say that this answer is written *in terms of* π. Otherwise, simplify the expression using a numerical approximation for π. Use 3.14 for π in this unit unless otherwise indicated.



My Notes

4. Complete the table by finding the circumference in terms of π for diners at the specified distances in feet from the center of the restaurant. Also find the exact distances (in terms of π) and approximate distances traveled for diners when the restaurant rotates 60°.

Radius (feet)	Circumference (feet)	Distance Traveled During a 60° Rotation (feet)
50		
40		
30		
20		
10		
1		

5. Describe any pattern in the exact distance traveled.

The ***arc length*** is the length of a portion of the circumference of a circle. The arc length is determined by the radius of the circle and by the angle measure that defines the corresponding arc, or portion, of the circumference.

6. **Model with mathematics.** Write a formula that represents the arc length s of a 60° angle with a radius r. Describe the relationship between s and r.

MATH TIP

The variable r is used to represent radius in formulas. The variable s is often used to represent distance.

My Notes

Check Your Understanding

7. Identify the *constant of proportionality* in the formula in Item 6.

8. Use the formula in Item 6 to find the approximate distance a diner will travel when seated at each of the following distances from the center of the restaurant.
a. 12 feet **b.** 38 feet

9. How far has a diner, seated 25 feet from the restaurant center, traveled after rotating 120°? Explain how you found your answer.

MATH TIP

Recall that in the direct variation equation $y = kx$, x and y are proportional and k is the *constant of proportionality*.

10. Find the exact distances (in terms of π) and the approximate distances traveled by diners seated at the indicated distances from the center after the restaurant rotates 90°. Fill in the table.

Radius (feet)	Distance Traveled During a 90° Rotation (feet)
10	
20	
30	
40	
50	

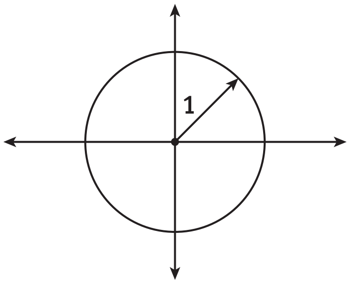
11. Reason quantitatively. Write a formula that represents the arc length s generated by a radius r that rotates 90°. Compare and contrast this with the formula you wrote in Item 6.

12. In Item 9, you found the length of the arc s generated by the 120° rotation of a 25-foot radius r. What is the constant of proportionality in a formula that defines s in terms of r for 120°? Give an exact answer in terms of π.

My Notes

As you can see, the constant of proportionality used to find arc length s in terms of radius r is different for each angle of rotation.

When you find the arc length generated by a radius on a circle with radius 1, called a ***unit circle***, you will find that the constant of proportionality takes on additional meaning.



13. Model with mathematics. Write a formula for s in terms of r on a unit circle when the angle of rotation is 180°. Identify the constant of proportionality. Also identify the value of s.

MATH TERMS

The angle of rotation is measured in degrees or **radians**. An angle's measurement in radians equals the length of a corresponding arc on the unit circle. Radian measures are often written in terms of π.

On a unit circle, the constant of proportionality is the measure of the angle of rotation written in ***radians***, which equals the length of the corresponding arc on the unit circle. For example, we say that 180° equals π radians. We can use this fact about the relationship between s and r on the unit circle to convert degree measures to radian measures. It may be helpful to write these as proportions.

14. Convert each degree measure to radians. Give the answers in terms of π.

a. 30° **b.** 45° **c.** 360°

Check Your Understanding

15. A circle has a radius of 15 feet. What is the length of the arc generated by a 45° angle?

16. What is the arc length generated by the 20° angle rotation on a circle that has a radius of 35 inches?

17. Convert each degree measure to radians.

a. 135° **b.** 120° **c.** 270°

LESSON 31-1 PRACTICE

18. What is the length of the arc formed by a 90° angle on a circle with a radius of 68 feet?

19. Attend to precision. What is the constant of proportionality for each angle measure? Write each answer in terms of π.
a. 40° **b.** 225°

20. Find the length of an arc formed by a 75° angle on a circle with a radius of 35 feet. Give the answer in terms of π.

21. Convert each degree measure to radians.
a. 35° **b.** 80°

Use the following information for Items 22–23. A diner has a circular dessert case in which the shelves inside rotate, but pause at set increments. Yesterday the restaurant manager decided to have the shelves pause every 60°.

22. How far did a lemon tart travel between each pause if it was placed on a shelf at a radius of 8 inches?

23. Express regularity in repeated reasoning. How far does a custard travel between each pause if it is placed at a radius of 12 inches?

My Notes

CONNECT TO AP

In calculus, all angles are assumed to be measured in radians.

My Notes

Learning Targets:

- Develop and apply formulas for the length of an arc.
- Apply radian measure.

SUGGESTED LEARNING STRATEGIES: Create a Plan, Look for a Pattern, Work Backward, Share and Respond, Create Representations

Angle measures can be given in degrees or radians. Angle measures in degrees are converted to radians to find arc length. Since we generally think of angles in degrees, it is useful to also know how to convert radian measures to degrees.

1. In Lesson 31-1, you found that $180° = \pi$ radians. What ratio can you multiply π radians by to convert it back to $180°$?

2. Does this ratio also help you convert $\frac{\pi}{2}$ radians to $90°$? Show how you determined your answer.

3. **Make use of structure.** How can you convert an angle measure given in radians to degrees?

4. Convert the following angles in radians to degrees.

a. $\frac{\pi}{5}$ **b.** $\frac{\pi}{4}$ **c.** $\frac{3\pi}{2}$

Sometimes angles greater than $360°$ are also given in radians.

5. Convert the following angles in radians to degrees.

a. $\frac{7\pi}{3}$ **b.** $\frac{6\pi}{2}$ **c.** $\frac{11\pi}{4}$

My Notes

6. Given an angle in radian measure, how can you determine if the degree measure is less than or greater than 180° before doing the conversion?

7. Given an angle in radian measure, how can you tell if the degree measure is greater than 360° before you do the conversion?

Check Your Understanding

8. Convert the following angles in radians to degrees.
   a. $\frac{7\pi}{4}$ **b.** $\frac{8\pi}{3}$
9. **a.** Is $\frac{6\pi}{5}$ radians greater than or less than 180°? Than 360°?
   b. Convert $\frac{6\pi}{5}$ radians to degrees.
10. **Construct viable arguments.** Before converting, how can you tell if a radian angle measure will be between 180° and 360°?

Let's think about the rotating restaurant from Lesson 31-1. You concluded that the distance traveled by a diner in the restaurant could be found using $S = \left(\frac{\pi}{3}\right)r$ for a 60° angle. You also now know that $\frac{\pi}{3}$ radian is equal to 60°.

11. **Express regularity in repeated reasoning.** Write a formula to find arc length s traveled by a diner in the restaurant for any radian angle measure θ and any radius r.

MATH TIP

The Greek symbol theta (θ) is often used to represent an angle measure in a formula.

The designers decide that the restaurant should do one complete rotation every 40 minutes.

12. Approximately how far will a diner seated at a radius of 20 feet travel after dining for 1 hour, 20 minutes?

My Notes

13. Approximately how far will a diner seated at a radius of 50 feet travel after dining for 1 hour 20 minutes?

Check Your Understanding

14. How far will a diner seated 10 feet from the center of the restaurant travel in 1 hour?

15. How far will a diner seated 50 feet from the center travel in 1 hour?

16. How long does it take a diner seated 50 feet from the center to travel the distance that the diner seated 10 feet from the center travels in 1 hour?

LESSON 31-2 PRACTICE

17. **Reason quantitatively.** Convert the following radians to degrees.

a. $\frac{4\pi}{5}$ **b.** $\frac{3\pi}{4}$ **c.** $\frac{5\pi}{3}$

18. A diner in a rotating restaurant is seated and travels $\frac{2\pi}{5}$ radians before the waiter comes to the table. How many degrees does he travel before the waiter arrives?

19. A rotating dessert case does a full rotation every 9 minutes. How far will a dessert item travel in 30 minutes if placed at a radius of 6 inches?

20. The dessert case in Item 19 is sped up so that it does a complete rotation every 5 minutes. How far will a piece of dessert travel in 15 minutes if placed at a radius of 9 inches?

21. **Critique the reasoning of others.** Kyle says the radian angle measure $\frac{5\pi}{2}$ is between 180° and 360°. Is he correct? Explain your thinking. How many degrees is $\frac{5\pi}{2}$ radians?

ACTIVITY 31 PRACTICE

Write your answers on notebook paper.
Show your work.

Lesson 31-1

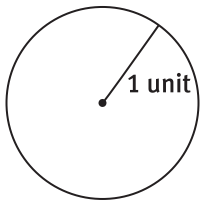
1. What is the approximate length of the arc formed by a 90° angle on a circle that has a radius of 70 feet?
A. 55 ft **B.** 110 ft
C. 220 ft **D.** 440 ft

2. A horse on a merry-go-round is positioned at a radius of 15 feet. How far will the horse travel after the merry-go-round rotates 60°?
A. 15.7 ft **B.** 23.6 ft
C. 31.4 ft **D.** 47.1 ft

3. A ticketholder is sitting on a bench that is on the merry-go-round. The ticketholder is sitting at a radius of 10 feet from the center. Approximately how far will the ticketholder travel after traveling 180° on the ride?

4. Several ticketholders are standing at various positions on the merry-go-round. Find the approximate distance ticketholders standing at the following radii will travel after the merry-go-round rotates 120°.
a. 11 feet **b.** 14 feet **c.** 16 feet

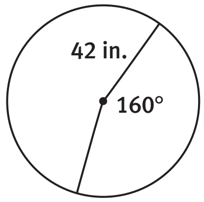
5. Use the unit circle. What is the constant of proportionality for each of the following angles? Give your answer in terms of π.



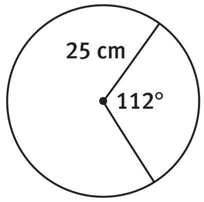
a. 24° **b.** 300°
c. 72° **d.** 270°

Find the arc lengths in Items 6 and 7.

6.



7.



8. Find the length of the arc formed by each angle and the given radius.
a. radius: 40 in., angle: 20°
b. radius: 12 m, angle: 90°
c. radius: 38 ft, angle: 75°

9. How many radians equal 225°?

10. Convert each degree measure to radians.
a. 48° **b.** 54°
c. 160° **d.** 120°

Lesson 31-2

11. Convert the following radian angle measures to degrees:

a. $\frac{\pi}{10}$ **b.** $\frac{5\pi}{6}$

c. $\frac{8\pi}{3}$ **d.** $\frac{7\pi}{4}$

e. $\frac{11\pi}{9}$ **f.** $\frac{10\pi}{3}$

g. $\frac{3\pi}{5}$ **h.** 4π

12. Is $\frac{\pi}{2}$ radians greater than, less than, or equal to 180°?

13. Is $\frac{3\pi}{4}$ radians greater than, less than, or equal to 180°?

14. Is $\frac{9\pi}{4}$ radians greater than, less than, or equal to 360°?

15. Is 2π radians greater than, less than, or equal to 360°?

16. A ticketholder on the merry-go-round is riding a horse that is at a radius of 12 feet. How far does she travel after the merry-go-round rotates $\frac{3\pi}{5}$ radians?

Use the following information for Items 17–20.
A merry-go-round makes one complete rotation every 80 seconds.

17. Approximately how far will a ticketholder seated at a radius of 15 feet travel after 60 seconds?

18. Approximately how far will a ticketholder standing at a radius of 16 feet travel after 140 seconds?

19. Approximately how far will a ticketholder seated at a radius of 12 feet travel after 110 seconds?

MATHEMATICAL PRACTICES
Reason Abstractly and Quantitatively

20. A ticketholder seated at a radius of 14 feet rode the merry-go-round for 120 seconds. Find the distance the ticketholder traveled. What is the measure of the angle over which the ticketholder rotated in degrees? Explain how you found your answer.

Trigonometric Functions

Which Angle is Up?

Lesson 32-1 Placing the Unit Circle on the Coordinate Plane

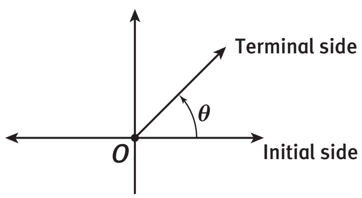
My Notes

Learning Targets:

- Explore angles drawn in standard position on the coordinate plane.
- Find the sine of θ and the cosine of θ.

SUGGESTED LEARNING STRATEGIES: Vocabulary Organizer, Close Reading, Create Representations, Sharing and Responding, Look for a Pattern

In the last lesson you worked with angles formed by radii within a circle. In trigonometry, we work with angles on the coordinate plane. An angle is in ***standard position*** when the vertex is placed at the origin and the ***initial side*** is on the positive x-axis. The other ray that forms the angle is the ***terminal side***.

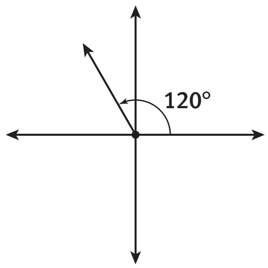


The terminal sides of angles with positive measures are formed by counterclockwise rotations. Angles with negative measures are formed by clockwise rotation of the terminal side.

Example A

Draw an angle in standard position with a measure of 120°.

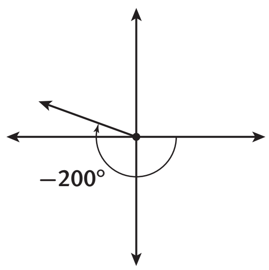
Since 120° is 30° more than 90°, the terminal side is 30° counterclockwise from the positive y-axis.



Example B

Draw an angle in standard position with a measure of −200°.

Since −200° is negative, the terminal side is 200° clockwise from the positive x-axis.

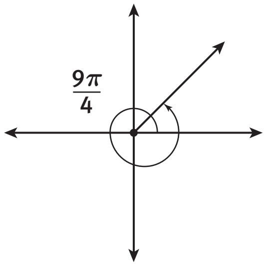


My Notes

Example C

Draw an angle in standard position with a measure of $\frac{9\pi}{4}$ radians.

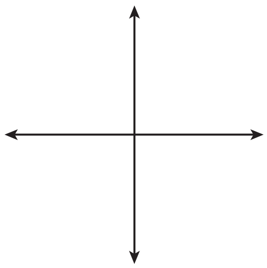
Since $\frac{9\pi}{4}$ is greater than 2π radians, the terminal side makes one full rotation, plus an additional $\frac{\pi}{4}$ radians.



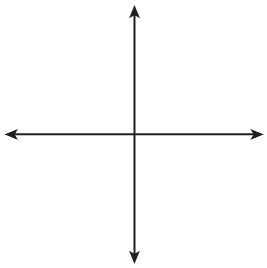
Try These A–C

Draw an angle in standard position with the given angle measure.

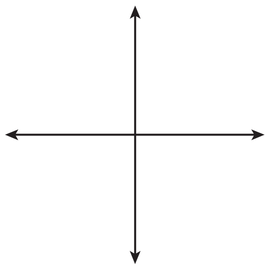
a. $290°$



b. $-495°$



c. $\frac{5\pi}{6}$



Angles can have different rotations but have the same initial and terminal sides. Such angles are ***coterminal angles***. In Example C, you can see that an angle that is $\frac{9\pi}{4}$ radians is coterminal with an angle that is $\frac{\pi}{4}$ radians.

1. How can you find an angle that is coterminal with a given angle, whether given in degrees or in radians?

My Notes

Example D

Find one positive and one negative angle that are coterminal with each given angle.

a. $225°$

$225° + 360° = 585°$

$225° - 360° = -135°$

b. $\frac{\pi}{3}$ radians

$\frac{\pi}{3} + 2\pi = \frac{7\pi}{3}$

$\frac{\pi}{3} - 2\pi = -\frac{5\pi}{3}$

Try These D

Find one positive and one negative angle that are coterminal with each given angle.

a. $150°$

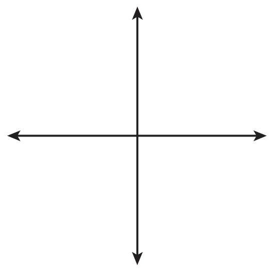
b. $320°$

c. $-270°$

d. $\frac{2\pi}{5}$

Check Your Understanding

2. Draw an angle in standard position with a measure of $\frac{10\pi}{3}$ radians.



3. Find one positive and one negative angle that are coterminal with each of the given angles.

a. $-330°$ **b.** $480°$ **c.** $\frac{3\pi}{2}$

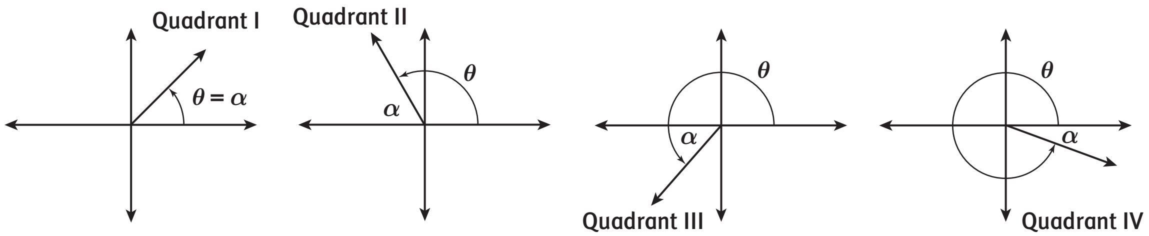
4. Are $520°$ and $-560°$ coterminal angles? Explain your answer.

5. Are $\frac{10\pi}{6}$ and $-\frac{28\pi}{6}$ coterminal angles? Explain your answer.

6. Is there a limit to the number of coterminal angles an angle can have? Explain.

My Notes

If θ is an angle in standard position, its ***reference angle*** α is the acute angle formed by the terminal side of θ and the x-axis. The graphs show the reference angle α for four different angles that have their terminal sides in different quadrants.



The relationship between θ and α is shown for each quadrant when $0° < \theta < 360°$ or $0 < \theta < 2\pi$.

	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
Degrees:	$\alpha = \theta$	$\alpha = 180° - \theta$	$\alpha = \theta - 180°$	$\alpha = 360° - \theta$
Radians:	$\alpha = \theta$	$\alpha = \pi - \theta$	$\alpha = \theta - \pi$	$\alpha = 2\pi - \theta$

Example E

Find the reference angle for $\theta = 245°$.

The terminal side of θ lies in Quadrant III.

$\alpha = 245° - 180°$, so $\alpha = 65°$.

Example F

Find the reference angle for $\theta = \frac{3\pi}{4}$.

The terminal side of θ lies in Quadrant II.

$\alpha = \pi - \frac{3\pi}{4}$, so $\alpha = \frac{\pi}{4}$.

When an angle is not between 0 and $360°(2\pi)$, find a coterminal angle that is within that range. Then use the coterminal angle to find the reference angle.

Example G

Find the reference angle for $\theta = 435°$.

Since $435°$ is greater than $360°$, subtract.

$435 - 360 = 75°$

Now determine the reference angle for $75°$.

Since $75°$ is in Quadrant I, the reference angle is $75°$.

My Notes

Example H

Find the reference angle for $\theta = \frac{13\pi}{4}$ radians.

Since $\frac{13\pi}{4}$ is greater than 2π, subtract.

$\frac{13\pi}{4} - 2\pi = \frac{13\pi}{4} - \frac{8\pi}{4} = \frac{5\pi}{4}$

The terminal side of this angle is in Quadrant III.

$\alpha = \frac{5\pi}{4} - \pi = \frac{5\pi}{4} - \frac{4\pi}{4}$, so $\alpha = \frac{\pi}{4}$.

Try These E–H

Find the reference angle for each value of θ.

a. $\theta = 325°$

b. $\theta = \frac{4\pi}{9}$

c. $\theta = 515°$

d. $\theta = \frac{7\pi}{6}$

e. $\theta = \frac{10\pi}{3}$

f. $\theta = 820°$

Coterminal and reference angles can be used to find trigonometric functions for angles in standard form. ***Trigonometric functions*** describe the relationships between sides and angles in a triangle.

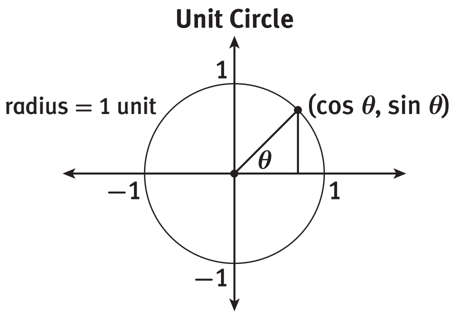
We can look at these relationships on the unit circle. Since the radius of the unit circle is 1 unit, any right triangle with a hypotenuse formed by this radius has a hypotenuse length of 1 unit.

MATH TIP

You learned the trigonometric functions for sine and cosine in geometry.

$\sin\theta = \frac{\text{opposite leg}}{\text{hypotenuse}}$

$\cos\theta = \frac{\text{adjacent leg}}{\text{hypotenuse}}$



$\sin\theta = \frac{y}{1} = y$

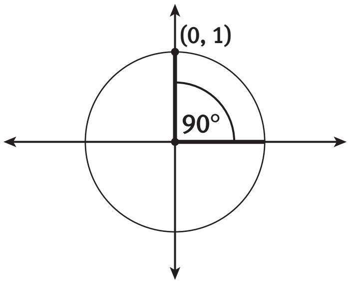
$\cos\theta = \frac{x}{1} = x$

My Notes

The cosine of θ ($\cos\theta$) is the x-coordinate of the point at which the terminal side of the angle intersects the unit circle. The sine of θ ($\sin\theta$) is the y-coordinate.

Example I

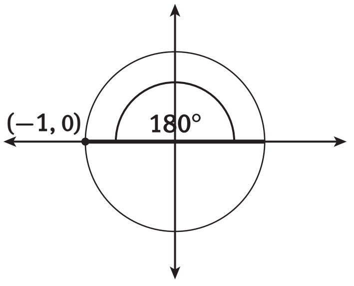
Find the sine and cosine of 90°.



$\sin 90° = 1$
$\cos 90° = 0$

Example J

Find the sine and cosine of 180°.



$\sin 180° = 0$
$\cos 180° = -1$

Try These I–J

a. What are the $\sin\theta$ and $\cos\theta$ for $\theta = 270°$, $\theta = -270°$, and $\theta = 720°$?

b. What are the $\sin\theta$ and $\cos\theta$ for $\theta = \pi$, $\theta = 2\pi$, and $\theta = -\frac{\pi}{2}$?

My Notes

Check Your Understanding

7. Find the reference angle for each value of θ.
  a. $\theta = 135$ **b.** $\theta = 240°$
  c. $\theta = \frac{7\pi}{6}$ **d.** $\theta = \frac{5\pi}{3}$

8. Find the value of sin θ and cos θ for each angle.
  a. $\theta = 360°$ **b.** $\theta = -90°$ **c.** $\theta = -\frac{7\pi}{2}$

LESSON 32-1 PRACTICE

9. Draw an angle in standard position with a measure of $-\frac{7\pi}{3}$ radians.

10. Give one positive and one negative angle that are coterminal with $-390°$.

11. What is the reference angle for each value of θ?
  a. $\theta = \frac{17\pi}{6}$ **b.** $\theta = -250°$?

12. What are the sine and cosine for each value of θ?
  a. $\theta = 270°$ **b.** $\theta = -5\pi$

13. Attend to precision. Refer to Examples I and J and Try These I–J. Do you notice anything about the sine and cosine of angles that are multiples of 90°?

My Notes

Learning Targets:

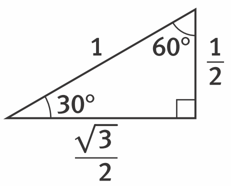
- Find the sine of θ and the cosine of θ using special right triangles.
- Find the tan of θ.

SUGGESTED LEARNING STRATEGIES: Create Representations, Graphic Organizer, Look for a Pattern

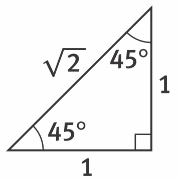
You can use what you know about the ratios of side lengths of special right triangles to determine the sine and cosine of their angles. As shown in the last lesson, a right triangle whose hypotenuse is a radius of the unit circle has a hypotenuse length of 1 unit. The hypotenuse is the terminal side of an angle, θ, and the sine and cosine of θ are the lengths of the legs of the right triangle.

MATH TIP

The ratio of the side lengths of a 30°-60°-90° triangle is $1 : \sqrt{3} : 2$, and of a 45°-45°-90° triangle is $1 : 1 : \sqrt{2}$. If the length of the hypotenuse of a 30°-60°-90° triangle is equal to 1, then the ratio must be divided by 2 to find the lengths of the legs, $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$.



If the length of the hypotenuse of a 45°-45°-90° triangle is 1, then the ratio must be divided by $\sqrt{2}$ to find the length of both legs, $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

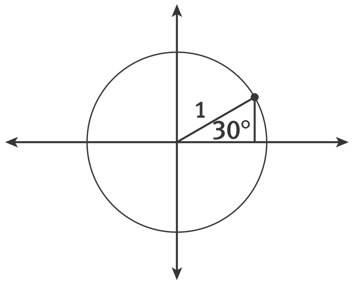


Example A

What are the sine and cosine of θ?

$\theta = 30°$

The sine and cosine are the lengths of the legs of a 30°-60°-90° triangle.



$\sin 30° = y = \text{length of shorter leg} = \frac{1}{2}$

$\cos 30° = x = \text{length of longer leg} = \frac{\sqrt{3}}{2}$

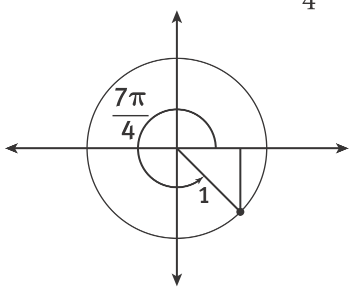
If θ is not in the first quadrant, use a reference angle.

Example B

What are $\sin\theta$ and $\cos\theta$?

$\theta = \frac{7\pi}{4}$ radians

To find $\sin\theta$ and $\cos\theta$, draw the terminal side of the angle on the unit circle. Make a right triangle with one leg on the x-axis. Determine the reference angle, which is $\frac{\pi}{4}$, or 45°. The triangle is a 45°–45°–90° triangle.



$\sin\frac{7\pi}{4} = y = -\text{length of opposite leg} = -\frac{\sqrt{2}}{2}$

$\cos\frac{7\pi}{4} = x = \text{length of adjacent leg} = \frac{\sqrt{2}}{2}$

My Notes

Try These A–B

What are sin θ and cos θ for each value of θ?

a. $\theta = 300°$

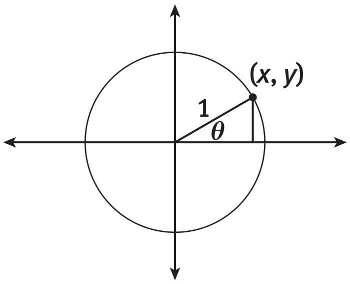
b. $\theta = -225°$

c. $\theta = \frac{5\pi}{6}$

d. $\theta = -\frac{4\pi}{3}$

Sine and cosine are just two of the trigonometric functions. Next we will look at a third function, the tangent function.

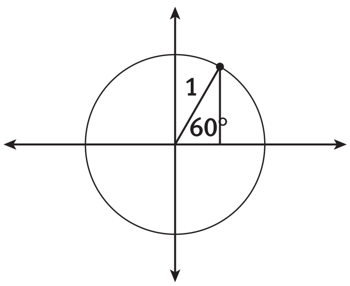
Recall that the tangent function for a right triangle is $\tan\theta = \frac{\text{opposite leg}}{\text{adjacent leg}}$. Looking at the unit circle on the coordinate plane, you can see that this can also be expressed as $\tan\theta = \frac{y}{x}$, where y and x are the coordinates at the point of intersection of the terminal side of θ and the unit circle.



As with the relationships we saw with sine and cosine, this relationship is also true for all angles on the unit circle.

Example C

What is tan θ for $\theta = 60°$?



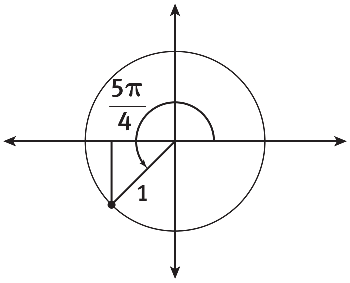
$$\tan 60° = \frac{y}{x} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

My Notes

Example D

What is $\tan\theta$ for $\theta = \frac{5\pi}{4}$?

Use the reference angle $\frac{\pi}{4}$.



$$\tan\frac{5\pi}{4} = \tan\frac{\pi}{4} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

Try These C–D

Find $\tan\theta$ for each value of θ.

a. $\theta = 300^\circ$

b. $\theta = 450^\circ$

c. $\theta = \frac{2\pi}{3}$

d. $\theta = \frac{11\pi}{4}$

MATH TIP

When a ratio has a denominator of 0, the ratio is *undefined*.

MATH TIP

When a ratio has an irrational number in the denominator, the denominator needs to be rationalized.

Multiply the numerator and denominator by the irrational number.

For example, $\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.

Check Your Understanding

1. Find $\sin\theta$ and $\cos\theta$.
a. $\theta = 210^\circ$ **b.** $\theta = \frac{2\pi}{3}$ **c.** $\theta = -\frac{\pi}{4}$

2. Find $\tan\theta$ for each value of θ.
a. $\theta = 240^\circ$ **b.** $\theta = 690^\circ$ **c.** $\theta = -585^\circ$

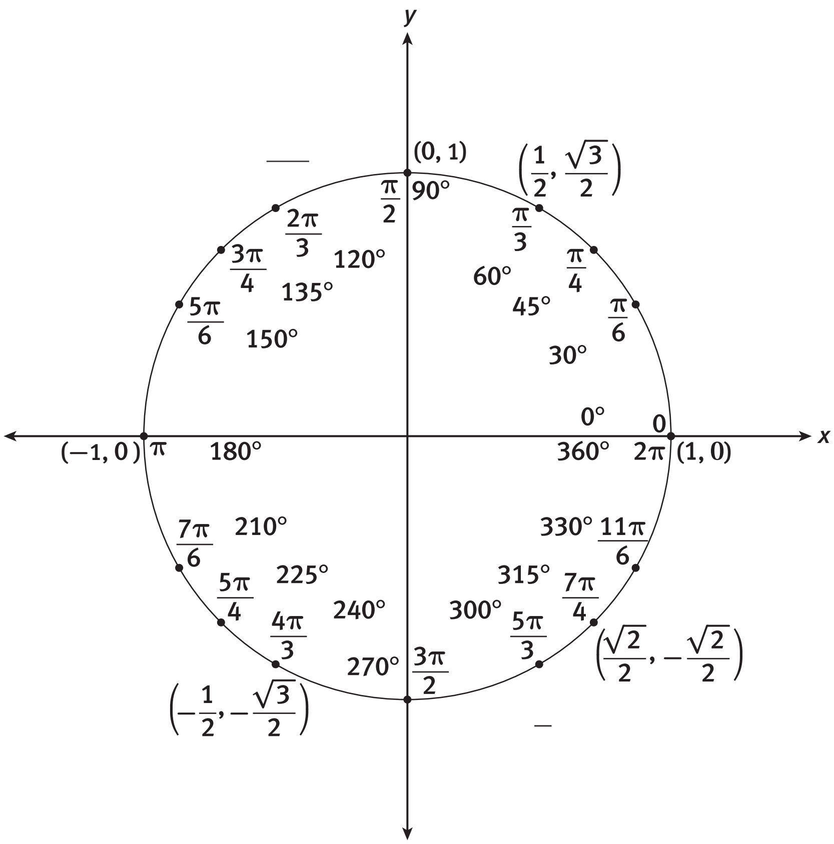
3. What is $\tan\theta$ for these values of θ?
a. $\theta = \frac{7\pi}{6}$ **b.** $\theta = \frac{7\pi}{3}$ **c.** $\theta = -\frac{9\pi}{4}$

My Notes

The terminal side of every angle in standard position has a point that intersects the unit circle. You have seen that a right triangle can be drawn with the terminal side of each angle as the hypotenuse. One leg of the triangle is the segment drawn from the point of intersection to the x-axis, and the other leg is the segment of the x-axis from the origin to the point of intersection with the vertical segment.

You have been looking at 30°-60°-90° triangles and 45°-45°-90° triangles. All of the angles that can form these two triangles are given on the unit circle below in degrees and radians.

4. Use the reference angle that can be formed to find the x- and y-coordinates for each point of intersection on the unit circle.



As you have seen in Lesson 32-1 and in the first part of this lesson, you can find the values of the trigonometric functions sine, cosine, and tangent using the coordinates of the point of intersection of the terminal side of each angle with the unit circle.

5. Use the coordinates you found in Item 4. What are the sine, cosine, and tangent of 210°?

6. What are the sine, cosine, and tangent of $\frac{5\pi}{4}$ radians?

MATH TIP

The coordinates of the intersection of the terminal side of an angle θ with the unit circle are $(\cos\theta, \sin\theta)$.

My Notes

Check Your Understanding

7. What are the sine, cosine, and tangent of $495°$?

8. What are the sine, cosine, and tangent of $\frac{7\pi}{4}$ radians?

LESSON 32-2 PRACTICE

9. Find the sine and cosine for each value of θ.

a. -300 **b.** $\frac{8\pi}{3}$

10. What is the point of intersection of the terminal side of $-120°$ with the unit circle?

11. What is the point of intersection of the terminal side of $\frac{5\pi}{2}$ with the unit circle?

12. Find the sine, cosine, and tangent of each angle.

a. $780°$ **b.** $-150°$ **c.** $-405°$

13. Find the sine, cosine, and tangent of each angle.

a. $-\frac{11\pi}{4}$ **b.** $-\frac{7\pi}{6}$ **c.** $\frac{10\pi}{3}$

14. **Make sense of problems.** Look at the unit circle in Item 4. Is it possible to have a negative value for sine but a positive value for tangent? Explain.

ACTIVITY 32 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 32-1

1. Draw an angle in standard position for each of the following measures.
   a. $200°$ b. $575°$
   c. $-225°$ d. $-660°$
   e. $\frac{2\pi}{5}$ f. $-\frac{3\pi}{2}$
   g. $-\frac{9\pi}{4}$ h. $\frac{11\pi}{3}$

2. Which angle is a coterminal angle with $140°$?
   A. $-140°$ B. $40°$
   C. $400°$ D. $500°$

3. Which angle is a coterminal angle with $-75°$?
   A. $435°$ B. $-285°$
   C. $285°$ D. -645

4. Which angle is *not* a coterminal angle with $\frac{5\pi}{4}$ radians?
   A. $-\frac{3\pi}{4}$ B. $-\frac{7\pi}{4}$
   C. $-\frac{11\pi}{4}$ D. $\frac{13\pi}{4}$

5. Give one positive and one negative angle that are coterminal with each of the following angles.
   a. $-65°$ b. $500°$
   c. $-\frac{6\pi}{5}$ d. $\frac{8\pi}{3}$

6. What is the reference angle for $\theta = 75°$?
   A. $15°$ B. $75°$
   C. $105°$ D. $255°$

7. What is the reference angle for $\theta = \frac{8\pi}{5}$?
   A. $\frac{\pi}{5}$ B. $\frac{2\pi}{5}$
   C. $\frac{3\pi}{5}$ D. $\frac{8\pi}{5}$

8. What is the reference angle for each value of θ?
   a. $\theta = -325°$ b. $\theta = 530°$
   c. $\theta = -\frac{12\pi}{5}$ d. $\theta = \frac{7\pi}{4}$

9. In which quadrant is the reference angle α equal to θ?

10. Find $\sin\theta$ and $\cos\theta$.
    a. $\theta = -180°$ b. $\theta = 450°$?

11. Find $\sin\theta$ and $\cos\theta$.
    a. $\theta = 6\pi$ b. $\theta = -\frac{7\pi}{2}$

12. What are the sine and cosine for each value of θ?
    a. $\theta = 315°$ b. $\theta = -510°$
    c. $\theta = -\frac{11\pi}{6}$ d. $\theta = \frac{10\pi}{3}$

Lesson 32-2

13. What is $\tan\theta$ for $\theta = -300°$?

A. $-\frac{\sqrt{3}}{3}$ **B.** $\frac{\sqrt{3}}{2}$

C. $\frac{1}{2}$ **D.** $\sqrt{3}$

14. What is $\tan\theta$ for $\theta = \frac{19\pi}{6}$?

A. $-\sqrt{3}$ **B.** $\frac{\sqrt{3}}{3}$

C. $\frac{\sqrt{3}}{2}$ **D.** $-\frac{1}{2}$

15. What is $\tan\theta$ for $\theta = 765°$?

A. $\sqrt{2}$ **B.** $\frac{\sqrt{2}}{2}$

C. -1 **D.** 1

16. What is $\tan\theta$ for each value of θ?

a. $\theta = -495°$ **b.** $\theta = 690°$

c. $\theta = \frac{14\pi}{3}$ **d.** $\theta = -\frac{7\pi}{2}$

17. Give an angle measure in degrees, between 0° and 360°, whose terminal side has a point of intersection with the unit circle at $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

18. Give an angle measure in radians, between π and 2π, whose terminal side has a point of intersection with the unit circle at $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

19. What are the sine, cosine, and tangent of 390°?

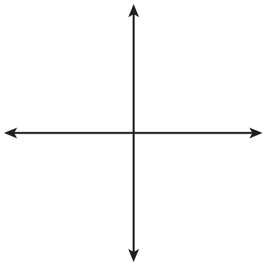
20. What are the sine, cosine, and tangent of −510°?

21. What are the sine, cosine, and tangent of $\frac{13\pi}{3}$?

22. What are the sine, cosine, and tangent of $-\frac{11\pi}{4}$?

MATHEMATICAL PRACTICES
Use Appropriate Tools Strategically

23. Use the unit circle in Item 4 of Lesson 32-2. Determine which trigonometric functions are positive and which are negative in each quadrant. Explain how you determined the signs for each quadrant. Summarize your findings on a coordinate plane like the one below.



Trigonometric Identities: Pythagorean Connection

More Than Just Triangles

Lesson 33-1 The Pythagorean Identity

My Notes

Learning Targets:

- Prove the Pythagorean identity.
- Use the Pythagorean identity to find sin θ, cos θ, or tan θ, given the value of one of these functions and the quadrant of θ.

SUGGESTED LEARNING STRATEGIES: Close Reading, Look for a Pattern, Discussion Groups, Create Representations

The trigonometric functions of sine, cosine, and tangent are each a ratio relating two of the three sides of a right triangle. Any two of these trigonometric ratios have one side in common, and together they relate all three sides of a triangle.

We can use the definitions of sine, cosine, and tangent to explore these relationships.

Look at the ratios that were defined in the previous lesson for the unit circle, where the length of the hypotenuse is equal to 1:

$\sin\theta = \frac{y}{1}$ $\qquad \cos\theta = \frac{x}{1}$ $\qquad \tan\theta = \frac{y}{x}$

Since $\sin\theta = y$ and $\cos\theta = x$, we can write the tan θ in terms of sine and cosine.

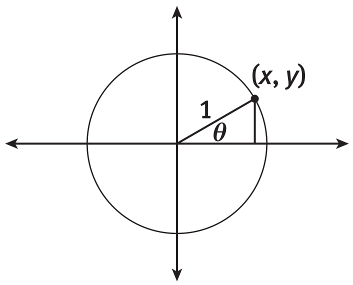
$$\tan\theta = \frac{y}{x}, \text{ so } \tan\theta = \frac{\sin\theta}{\cos\theta}$$

In geometry, you studied a special relationship between the sides of a right triangle when you learned the Pythagorean Theorem.

Let's express the relationship between the sides of a triangle on the unit circle with the Pythagorean Theorem.

MATH TIP

The Pythagorean Theorem shows the following relationship between the sides of a right triangle: $a^2 + b^2 = c^2$, where a and b are the legs and c is the hypotenuse of a right triangle.



Here we can see that the legs are x and y and the hypotenuse is 1, so $x^2 + y^2 = 1^2$. Simplified, $x^2 + y^2 = 1$.

My Notes

We can rewrite this equation with sine and cosine by substituting sin θ for y and cos θ for x. Now we have the following equation:

$$(\sin\theta)^2 + (\cos\theta)^2 = 1$$

Using the notation $\sin^2\theta$ for $(\sin\theta)^2$ and $\cos^2\theta$ for $(\cos\theta)^2$, this equation can be rewritten as follows:

$$\sin^2\theta + \cos^2\theta = 1$$

This relationship is called a Pythagorean identity.

We can use all of these relationships between sine, cosine, and tangent to solve problems on the unit circle.

MATH TIP

The sign of each trigonometric function depends on the quadrant in which the terminal side of the angle lies.

Example A

Given that $\cos\theta = -\frac{3}{5}$ and that $\frac{\pi}{2} < \theta < \pi$, find the value of $\sin\theta$ and $\tan\theta$.

Since we need $\sin\theta$ to calculate $\tan\theta$, let's first find $\sin\theta$.

Using $\sin^2\theta + \cos^2\theta = 1$, substitute any given information and solve.

$$\sin^2\theta + \left(-\frac{3}{5}\right)^2 = 1$$

$$\sin^2\theta + \left(\frac{9}{25}\right) = 1$$

$$\sin^2\theta = \frac{16}{25}$$

$$\sqrt{\sin^2\theta} = \sqrt{\frac{16}{25}}$$

$$\sin\theta = \frac{4}{5}$$

Because it is in the second quadrant, sine is positive.

Now we can find $\tan\theta$ using $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

$$\tan\theta = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3}$$

Try These A

a. Given that $\cos\theta = -\frac{8}{17}$ and that $\frac{\pi}{2} < \theta < \pi$, find the value of $\sin\theta$ and $\tan\theta$.

b. Given that $\cos\theta = -\frac{5}{13}$ and that $\pi < \theta < \frac{3\pi}{2}$, find the value of $\sin\theta$ and $\tan\theta$.

My Notes

Check Your Understanding

1. Given that $\cos\theta = \frac{7}{25}$ and that $0 < \theta < \frac{\pi}{2}$, find the value of $\sin\theta$ and $\tan\theta$.

2. Given that $\sin\theta = \frac{3}{5}$ and that $\frac{\pi}{2} < \theta < \pi$, find the value of $\cos\theta$ and $\tan\theta$.

LESSON 33-1 PRACTICE

3. Given that $\sin\theta = -\frac{40}{41}$ and that $\pi < \theta < \frac{3\pi}{2}$, find the value of $\cos\theta$ and $\tan\theta$.

4. Given that $\cos\theta = -\frac{5\sqrt{3}}{10}$ and that $\frac{\pi}{2} < \theta < \pi$, find the value of $\sin\theta$ and $\tan\theta$.

5. Given that $\sin\theta = -\frac{4\sqrt{2}}{8}$ and that $\frac{3\pi}{2} < \theta < 2\pi$, find the value of $\cos\theta$ and $\tan\theta$.

6. **Reason quantitatively.** If sine and cosine are both positive, in which quadrant is the terminal side of the angle?

7. **Reason abstractly.** When solving for a missing value of sine or cosine in the equation $\sin^2\theta + \cos^2\theta = 1$, is it possible that the answer may be negative? Explain.

My Notes

Learning Targets:

- Define the three reciprocal trigonometric functions.
- Use the Pythagorean identity and the reciprocal trigonometric functions to prove other trigonometric identities.

SUGGESTED LEARNING STRATEGIES: Think Aloud, Discussion Groups, Note Taking

In addition to sine, cosine, and tangent, there are three more trigonometric functions. These functions are secant (sec), cosecant (csc) and cotangent (cot). Each of these is a reciprocal of one of the first three trigonometric functions you have learned. Similarly, the first three can be considered reciprocals of the second three. The reciprocal identities are shown here.

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

1. In Lesson 33-1 you learned the tangent quotient identity, $\tan\theta = \frac{\sin\theta}{\cos\theta}$. Given that cotangent and tangent are reciprocals of one another, express $\cot\theta$ in terms of $\sin\theta$ and $\cos\theta$.

In Lesson 33-1 you also learned about one Pythagorean identity, $\sin^2\theta + \cos^2\theta = 1$. There are three Pythagorean identities altogether. You can use the reciprocal and quotient identities to find the other two.

Example A

Divide $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$ to find the second Pythagorean identity.

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

Simplify each ratio and substitute single trigonometric functions.

$$\tan^2\theta + 1 = \sec^2\theta$$

This is the second Pythagorean identity.

In a similar way, you can find the third Pythagorean identity.

Try These A

Divide $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$ to find the third Pythagorean identity.

My Notes

Check Your Understanding

2. Simplify $\frac{1}{\cos\theta}$.

3. Show how $\sec^2\theta = \frac{1}{\cos^2\theta}$ is equivalent to $\sec\theta = \frac{1}{\cos\theta}$.

In addition to these trigonometric identities, there are other forms of the identities that you can derive by multiplying one identity by a trigonometric function.

Example B

Multiply $1 + \cot^2\theta = \csc^2\theta$ by $\sin\theta$ to find another form of the trigonometric identity. Remember that if you multiply both sides of an equation by the same expression, it does not change the equation.

$1 + \cot^2\theta = \csc^2\theta$

$\sin\theta(1 + \cot^2\theta) = \sin\theta(\csc^2\theta)$

$\sin\theta + \sin\theta(\cot^2\theta) = \sin\theta\left(\frac{1}{\sin^2\theta}\right)$ Write parts you can in terms of sine.

$\sin\theta + \sin\theta\cot^2\theta = \csc\theta$ Simplify.

Other forms of trigonometric identities can be found this way as well.

Try These B

Make use of structure. Multiply $\tan^2\theta + 1 = \sec^2\theta$ by $\cos\theta$ to find another form of the trigonometric identity.

My Notes

Check Your Understanding

4. Multiply $\sin^2\theta + \cos^2\theta = 1$ by $\csc\theta$ to find another form of the trigonometric identity.

5. Divide $\sin^2\theta + \cos^2\theta = 1$ by $\cos\theta$ to find another form of the trigonometric identity.

LESSON 33-2 PRACTICE

5. Simplify $\frac{1}{\csc^2\theta}$.

6. **Make use of structure.** Write $\frac{1}{\tan\theta}$ in two other ways.

7. Multiply $1 + \cot^2\theta = \csc^2\theta$ by $\sin^2\theta$ to find another form of the trigonometric identity.

8. Multiply $\tan^2\theta + 1 = \sec^2\theta$ by $\sin^2\theta$ to find another form of the trigonometric identity.

9. **Critique the reasoning of others.** Danielle says multiplying by $\cos\theta$ is the same as dividing by $\csc\theta$. Is she correct? Explain your reasoning.

ACTIVITY 33 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 33-1

1. In which quadrant are sine, cosine, and tangent all positive?
   A. I **B.** II **C.** III **D.** IV

2. In which quadrant are both sine and cosine negative?
   A. I **B.** II **C.** III **D.** IV

3. Given that $\cos\theta = -\frac{11}{61}$ and that $\pi < \theta < \frac{3\pi}{2}$, what is the value of $\sin\theta$?
   A. $-\frac{11}{60}$ **B.** $\frac{11}{60}$
   C. $-\frac{60}{61}$ **D.** $\frac{60}{61}$

4. Given that $\cos\theta = \frac{15}{17}$ and that $\frac{3\pi}{2} < \theta < 2\pi$, find the value of $\sin\theta$ and $\tan\theta$.

5. Given that $\sin\theta = \frac{5}{13}$ and that $0 < \theta < \frac{\pi}{2}$, find the value of $\cos\theta$ and $\tan\theta$.

6. Given that $\sin\theta = -\frac{6\sqrt{2}}{12}$ and that $\frac{3\pi}{2} < \theta < 2\pi$, what is the value of $\cos\theta$?
   A. $\frac{6\sqrt{2}}{12}$ **B.** $-\frac{6\sqrt{2}}{12}$
   C. 1 **D.** -1

7. Given that $\sin\theta = \frac{2\sqrt{3}}{4}$ and that $\frac{\pi}{2} < \theta < \pi$, find the value of $\cos\theta$ and $\tan\theta$.

8. Given that $\cos\theta = -\frac{3\sqrt{2}}{6}$ and that $\pi < \theta < \frac{3\pi}{2}$, find the value of $\sin\theta$ and $\tan\theta$.

9. If the sine of an angle is positive and the cosine is negative, in which quadrant is the terminal side of the angle?
   A. I **B.** II **C.** III **D.** IV

Lesson 33-2

10. Simplify $\frac{1}{\csc\theta}$.

11. Which is $\frac{1}{\cos^2\theta}$ simplified?

A. $\sin^2\theta$ **B.** $\sec^2\theta$
C. $\csc^2\theta$ **D.** $\cot^2\theta$

12. Write $\frac{1}{\cot\theta}$ in two other ways.

13. Which expression(s) equal $\sin\theta$?

I. $\frac{1}{\csc\theta}$ III. $\frac{1}{\sec\theta}$
II. $\csc\theta$ IV. $\tan\theta\cos\theta$

A. I only
B. III only
C. I and IV
D. II and IV

14. Which expression(s) are not equal to $\tan^2\theta$?

I. $\sec^2\theta+1$ III. $\frac{\sin^2\theta}{\cos^2\theta}$
II. $\sec^2\theta-1$ IV. $\frac{1}{\cot^2\theta}$

A. I
B. II
C. III
D. I and IV

15. Which is the product of $\tan^2\theta+1=\sec^2\theta$ and $\cos^2\theta$?

A. $\csc^2\theta+\cos^2\theta=\cos^2\theta\sec^2\theta$
B. $\sec^2\theta+\cos^2\theta=\cot^2\theta$
C. $\sec^2\theta+\cos^2\theta=1$
D. $\sin^2\theta+\cos^2\theta=1$

16. Multiply $1+\cot^2\theta=\csc^2\theta$ by $\sin^2\theta$ to find another form of the trigonometric identity.

17. Multiply $\sin^2\theta+\cos^2\theta=1$ by $\csc^2\theta$ to find another form of the trigonometric identity.

18. What is the product of $\tan\theta$ and $\cot\theta$?

MATHEMATICAL PRACTICES

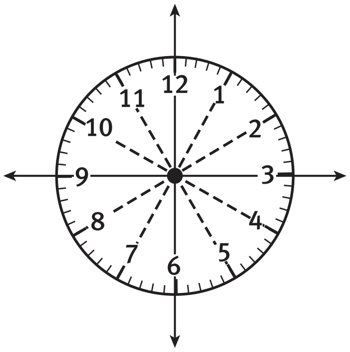
Make Sense of Problems and Persevere in Solving Them

19. Is $\sin^2\theta+\cos^2\alpha=1$ a true equation? Explain.

Radians, Unit Circles, and Trigonometry

A FLORAL CLOCK

A landscape architect is designing a large, circular garden that looks like a clock for a local park.



1. She wants to plant red dahlias in the sector between 7:00 and 9:00.
   - **a.** How many degrees would an hour hand travel between 7:00 and 9:00? How many radians? What portion of the circumference is the arc between 7:00 and 9:00?
   - **b.** The radius of the clock is 30 feet. How many linear feet of garden edging do they need for the sector containing dahlias?
   - **c.** Suppose the architect uses a unit circle for her plans. What are the coordinates of the point showing 7:00? Showing 9:00?
2. The architect wants to locate a stone rabbit on the edge of the clock wherever $x = \pm 0.6$. How many stone rabbits does she need? What are the coordinates of the locations?
3. Mahesh walked around the completed floral clock. He started at 3:00, walked counterclockwise around the clock three times, continued walking to 10:00, and then stopped.
   - **a.** How many degrees did he travel in all? How many radians?
   - **b.** When Mahesh walked around the clock along a fixed pathway, he was 45 feet from its center. How far did he walk altogether?
   - **c.** What angle between 0° and 360° is coterminal with his stopping place? What angle between 0 radians and 2π radians is coterminal with his stopping place?
   - **d.** What is the reference angle of his stopping place in degrees? What is it in radians?

When Evan walked into his math class, the teacher announced that scientific calculators would not be allowed on the trigonometry exam. After thinking about it, Evan realized he could use what he already knew to find the value of the trigonometric functions.

4. How can he use 45°-45°-90° and 30°-60°-90° triangles to figure out the values $\sin 45^\circ$ and $\cos 60^\circ$?
5. Evan used special right triangles to make a chart of the values of trigonometric functions for angles from 0° to 90°. Then he encountered a problem asking for $\tan \frac{\pi}{6}$. Explain how he can figure out its value.
6. Explain how he can use his chart to find the following values.
   - **a.** $\sin(-45^\circ)$
   - **b.** $\cos \frac{2\pi}{3}$
   - **c.** $\tan \frac{7\pi}{6}$
   - **d.** $\sin 750^\circ$
7. Given $\sec 18^\circ \approx 1.05$ explain how he can use the reciprocal identities and Pythagorean identities to find the values of the other functions.
   - **a.** $\tan 18^\circ$
   - **b.** $\cot 18^\circ$
   - **c.** $\cos 18^\circ$
   - **d.** $\sin 18^\circ$
   - **e.** $\csc 18^\circ$

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
	The solution demonstrates these characteristics:			
Mathematics Knowledge and Thinking (Items 1-7)	• Fluency in working with circles and angles measured in degrees and radians • Effective understanding of the definitions of the trigonometric functions and the unit circle • Clear and accurate evaluation of trigonometric functions using the unit circle, special right triangles, and trigonometric identities	• Little difficulty in working with circles and angles measured in degrees and radians • Adequate understanding of the definitions of the trigonometric functions and the unit circle • Largely correct evaluation of trigonometric functions using the unit circle, special right triangles, and trigonometric identities	• Some difficulty in working with circles and angles measured in degrees and radians • Partial understanding of the definitions of the trigonometric functions and the unit circle • Partially correct evaluation of trigonometric functions using the unit circle, special right triangles, and trigonometric identities	• Significant difficulty in working with circles and angles measured in degrees and radians • Little or no understanding of the definitions of the trigonometric functions and the unit circle • Inaccurate or incomplete evaluation of trigonometric functions using the unit circle, special right triangles, and trigonometric identities
Problem Solving (Items 1, 2, 3)	• An appropriate and efficient strategy that results in a correct answer	• A strategy that may include unnecessary steps but results in a correct answer	• A strategy that results in some incorrect answers	• No clear strategy when solving problems
Mathematical Modeling / Representations (Items 1, 2, 3)	• Effective understanding of how angles in circles, including the unit circle, relate to a real-world scenario	• Largely correct understanding of how angles in circles, including the unit circle, relate to a real-world scenario	• Partial understanding of how angles in circles, including the unit circle, relate to a real-world scenario	• Incomplete or inaccurate understanding of how angles in circles, including the unit circle, relate to a real-world scenario
Reasoning and Communication (Items 4, 5, 6, 7)	• Precise use of appropriate math terms and language when explaining how to evaluate trigonometric functions using a chart, the unit circle, or trigonometric identities	• Adequate use of math terms and language when explaining how to evaluate trigonometric functions using a chart, the unit circle, or trigonometric identities	• Misleading or confusing use of math terms and language when explaining how to evaluate trigonometric functions using a chart, the unit circle, or trigonometric identities	• Incomplete or mostly inaccurate use of math terms and language when explaining how to evaluate trigonometric functions using a chart, the unit circle, or trigonometric identities

Graphs of Trigonometric Functions

Creation of a Mural

Lesson 34-1 Periodic Functions

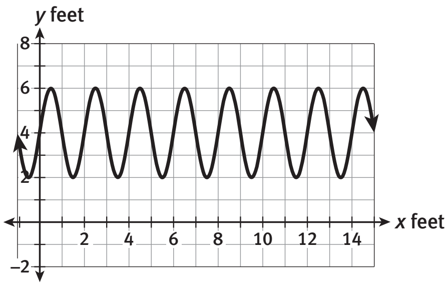
My Notes

Learning Targets:

- Identify periodic functions.
- Find the period, midline, and amplitude of periodic functions.

SUGGESTED LEARNING STRATEGIES: Close Reading, Paraphrasing, Create Representations, Vocabulary Organizer, Discussion Groups, Think-Pair-Share

An artist created this design to decorate a wall of the new transit center. The painters wondered if there is a mathematical description for the pattern to make it easier for them to reproduce it accurately.

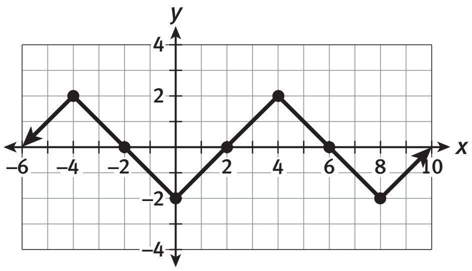


The pattern repeats at regular intervals, or *periods*, so it is called a ***periodic function***.

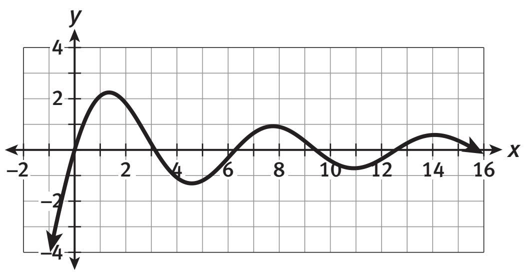
MATH TERMS

A **periodic function** is a function that repeats its values in regular intervals called periods.

This graph shows a periodic function. You can extend it in both directions by repeating its shape.



This graph does not show a periodic function. Although it extends in both directions, you cannot predict its shape, because it does not repeat at regular intervals.



Check Your Understanding

1. **Attend to precision.** Sketch the graph of a periodic function. Explain why it is periodic.
2. Sketch the graph of a function that is not periodic. Explain why it is not periodic.

My Notes

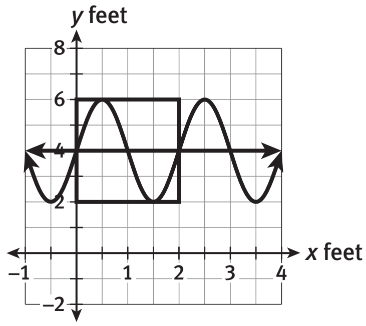
MATH TERMS

A **period** is the horizontal distance required for the graph of a periodic function to complete one repetition, or *cycle*.

The **amplitude** of a function is half the difference between the minimum and maximum values of the range.

The **midline** is a horizontal axis that is used as the reference line about which the graph of a periodic function oscillates.

To describe the design shown for the transit center wall more precisely, you can define its ***period***, ***amplitude***, and ***midline***. Study the graph shown here.

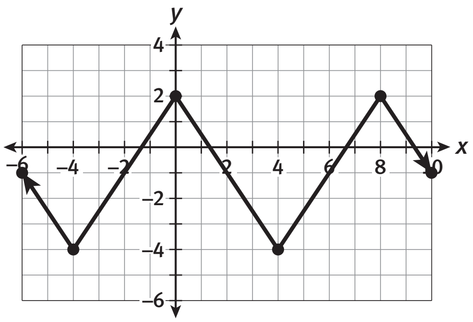


The portion of the design outlined by the rectangle shows one repetition, or *period*, of the function. The horizontal distance of one repetition is 2 units, so the period of the function is 2.

The graph oscillates between $y = 2$ and $y = 6$, so the range of the function is $2 \le y \le 6$. Half of that distance between these two values is called the *amplitude* of the function. Since $|6 - 2| \div 2 = 2$, the amplitude of the function is 2.

The horizontal line that runs midway between the maximum and minimum values of the function is the *midline*. Because 4 is midway between 6 and 2, the line $y = 4$ is the midline of the function.

3. Look at the following graph.



a. Draw a rectangle around exactly one repetition of the graph of the function.

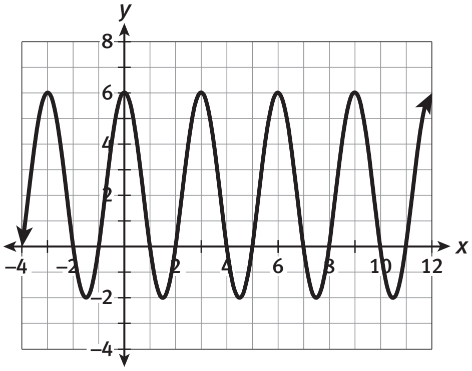
b. How wide is the rectangle?
What feature of the periodic function is the width of one repetition?

c. How high is the rectangle?
How can you use the height to find the amplitude?

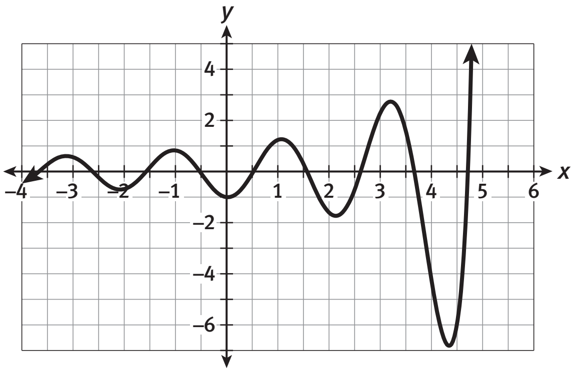
d. Draw the midline of the function.
What is the equation of the midline?

Decide if each graph shows a periodic function. If it does, give its period, its amplitude, and the equation of its midline. If it does not, explain why not.

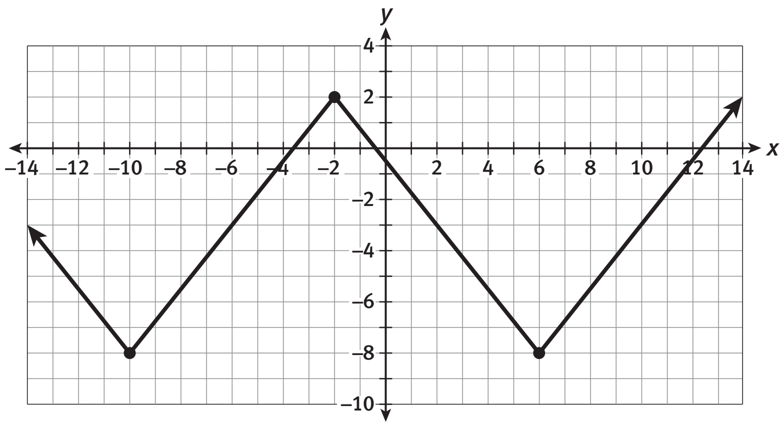
4.



5.



6.



MATH TIP

Sketch a rectangle around one repetition of the function. Study the portion of the graph in the rectangle to find the period, amplitude, and midline.

Check Your Understanding

7. Suppose you know the minimum and maximum values of a periodic function. How can you find its amplitude?

8. Reason quantitatively. How could you use the minimum and maximum values of a periodic function to find the equation of its midline?

MATH TIP

A *maximum* can be thought of as the greatest value of *y* or the *y*-value of the highest point on the graph.

A *minimum* is the least value of *y* or the *y*-value of the lowest point on the graph.

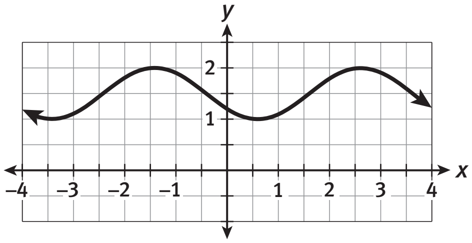
Periodic functions have more than one maximum and minimum.

My Notes

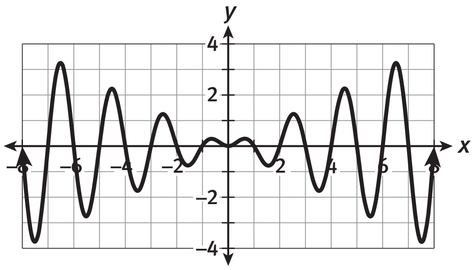
LESSON 34-1 PRACTICE

Decide if each graph shows a periodic function. If it does, give the period, the amplitude, and the equation of the midline. If it does not, explain why not.

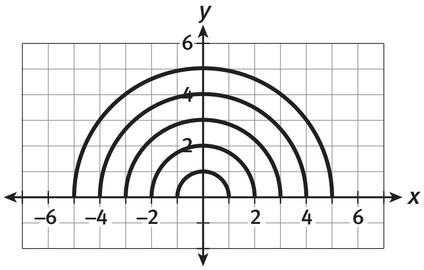
9.



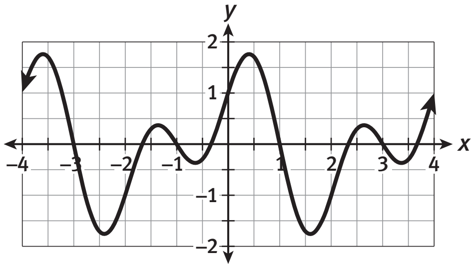
10.



11.



12.



13. Construct viable arguments. Samir says that when the amplitude of a periodic function doubles, the maximum value of the function doubles. Do you agree or disagree? Justify your response.

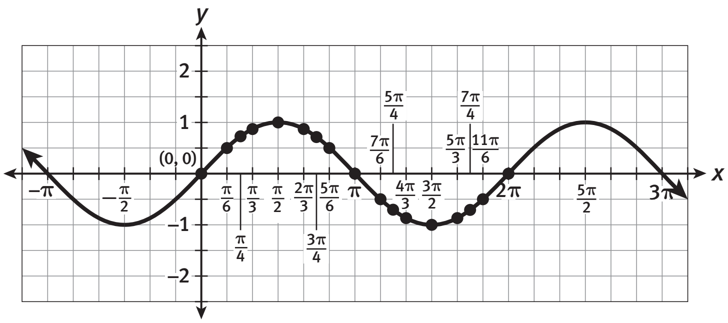
My Notes

Learning Targets:

- Graph the sine function, $y = a \sin b\,x$.
- Find the period, midline, and amplitude of sine functions.

SUGGESTED LEARNING STRATEGIES: Chunking the Activity, Close Reading, Paraphrasing, Create Representations, Discussion Groups, Think-Pair-Share

The sine function is an example of a periodic function. It repeats every 2π radians, or 360°. You can use a table of trigonometric values to plot the values you know and then connect them to show the graph of the sine function.



x	$\sin x$
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$
π	0
$\frac{7\pi}{6}$	$-\frac{1}{2}$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{3\pi}{2}$	-1
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{11\pi}{6}$	$-\frac{1}{2}$
2π	0

The graph of the sine function is symmetrical with respect to the origin, because it is unchanged when reflected across both the x- and y-axes. Therefore, it is an *odd function*. Another way to tell that it is an odd function is to see that the values for $\sin x$ and $\sin(-x)$ are additive inverses.

Check Your Understanding

1. Name the period, the midline, and the amplitude of the sine function.
2. **Express regularity in repeated reasoning.** If $\sin \frac{\pi}{2} = .2588$, what is the value of $\sin\left(\frac{-\pi}{2}\right)$? How do you know?
3. **Make use of structure.** How can you use coterminal angles to explain why the sine function between 2π and 4π radians is the same as it is between 0 and 2π radians?

MATH TIP

Recall that an *odd function* is symmetric with respect to the origin. For each x, $f(-x) = -f(x)$.

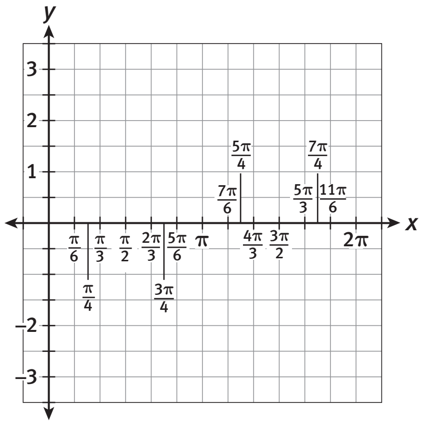
My Notes

TECHNOLOGY TIP

Be sure to set the calculator in radians before graphing in radians. Press MODE. Use the arrow keys to move the cursor over RADIAN, and press ENTER.

The parent sine function is $y = \sin x$. Changing the parent sine function transforms its graph. For example, a sine function may have a coefficient other than 1, written in the form $y = a \sin x$, where a is the coefficient of the function.

4. Use a graphing calculator to sketch and compare the graphs of $y = \sin x$, $y = 3 \sin x$, and $y = \frac{1}{3} \sin x$.



Notice that the periods, midlines, and x-intercepts are identical for all three graphs. However the amplitudes are different. The amplitude of $y = 3 \sin x$ is 3, and the graph is vertically stretched. The amplitude of $y = \frac{1}{3} \sin x$ is $\frac{1}{3}$ and the graph is vertically compressed. The amplitude of the function $y = a \sin \theta$ is $|a|$.

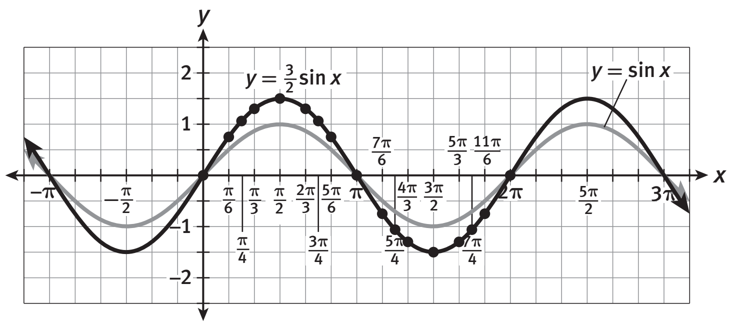
Example A

Draw the graph of $y = \frac{3}{2} \sin x$.

Name its period, amplitude, and midline.

Step 1: Lightly sketch the parent sine function.

Step 2: Find several key points on the curve. Multiply the y value of each point by $\frac{3}{2}$ and plot the new point. Connect the points.



The period is 2π, the amplitude is $\frac{3}{2}$, and the midline is $y = 0$.

Try These A

Graph each sine function on a separate coordinate plane. Name the period, amplitude, and midline for each.

a. $y = 2\sin x$ **b.** $y = \frac{1}{2}\sin x$ **c.** $y = -1\sin x.$

d. What is an equation of a sine function that has a period of 2π, an amplitude of 4, and a midline of $y = 0$?

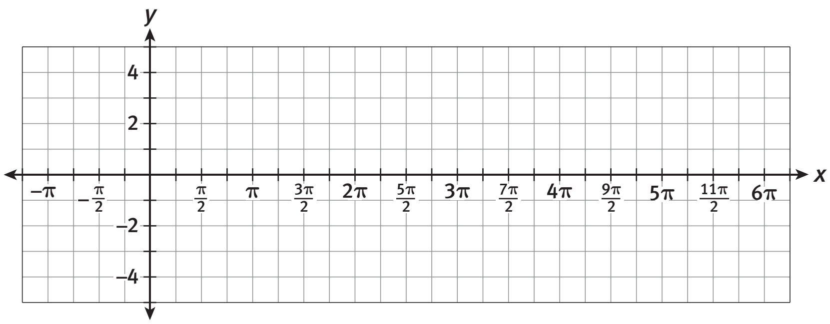
My Notes

TECHNOLOGY TIP

If the calculator does not show a full period, you can change the axes. Press WINDOW and change the *Xmin*, *Xmax*, *Xscl*, *Ymin*, *Ymax*, and *Yscl*, as needed.

The graph of the parent sine function is also transformed when the angle has a coefficient other than 1, written as $y = \sin bx$, where b is the coefficient of the angle.

5. Use appropriate tools strategically. Use a graphing calculator to compare the graphs of $y = \sin x$, $y = \sin 3x$, and $y = \sin \frac{1}{3}x$. Sketch and label the three graphs on the coordinate plane below.



Notice that the amplitudes and midlines are identical for all three graphs. The periods are different. The graph of $y = \sin 3x$ is compressed horizontally, so that it shows three cycles between 0 and 2π. It repeats every $\frac{2\pi}{3}$ units, so its period is $\frac{2\pi}{3}$. The graph of $y = \sin \frac{1}{3}x$ has $\frac{1}{3}$ of a cycle between 0 and 2π. It is stretched horizontally and shows one full cycle between 0 and 6π. Therefore, its period is 6π.

The period of the function $y = \sin bx$ is $\frac{2\pi}{b}$. It is found by dividing 2π by the coefficient of the angle.

Example B

Find the period of $y = \sin \frac{2}{3}x$. Sketch its graph.

Step 1: The coefficient of x is $\frac{2}{3}$. Simplify $\frac{2\pi}{\frac{2}{3}}$. The period is 3π. The graph completes one full cycle between 0 and 3π.

Step 2: Lightly sketch the parent sine function from 0 to 3π.

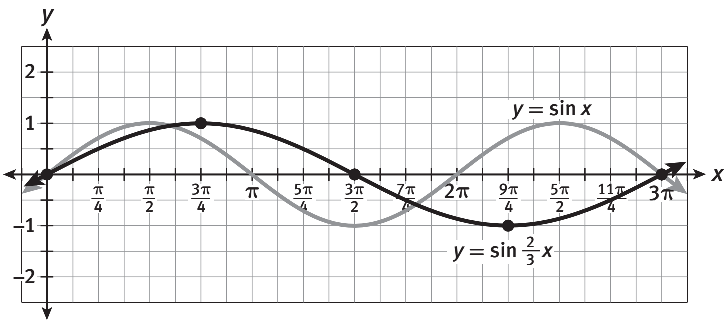
My Notes

Step 3: Plot points at 0 and 3π on the x-axis to show the beginning and end of one cycle of $y = \sin \frac{2}{3}x$.

Step 4: Plot a point at $\frac{3\pi}{2}$ on the x-axis. It is the halfway point of the cycle. It shows where the curve crosses the x-axis when going between the maximum and minimum.

Step 5: Plot a maximum or minimum point at $\frac{3\pi}{4}$ and at $\frac{9\pi}{4}$. These points are halfway between two zeros of the function.

Step 6: Connect the points with a smooth curve. Label the function.



Example C

Find the period of $y = \sin 4x$. Sketch its graph.

Step 1: The coefficient of x is 4. Simplify $\frac{2\pi}{4}$ to find that the period is $\frac{\pi}{2}$. That means the graph completes one full cycle between 0 and $\frac{\pi}{2}$.

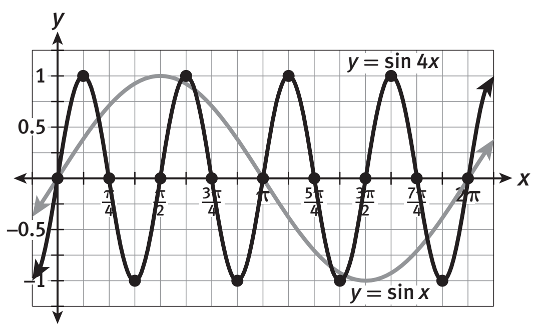
Step 2: Lightly sketch the parent sine function.

Step 3: Plot points at 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, and 2π on the x-axis to show four complete cycles of $y = \sin 4x$.

Step 4: Plot points on the x-axis halfway between those points to show where the curve crosses the x-axis when going between the maximum and minimum.

Step 5: Plot maximum or minimum points halfway between each two zeros of the function.

Step 6: Connect the points with a smooth curve. Label the function.



My Notes

Try These B–C

Model with mathematics. Find the period of each function. Graph at least one period of each function on a separate coordinate plane. Then state the amplitude and midline of each graph.

a. $y = \sin 2x$ **b.** $y = \sin \frac{1}{2}x$

c. What is an equation of a sine function that has a period of $\frac{\pi}{3}$, an amplitude of 1, and a midline of $y = 0$?

Check Your Understanding

6. What is the difference between the graphs of $y = 2\sin x$ and $y = \sin 2x$?

7. **Reason quantitatively.** If $0 < a < 1$, how does the graph of $y = a \sin x$ differ from the graph of $y = \sin x$?

8. If $0 < b < 1$, how does the graph of $y = \sin bx$ differ from the graph of $y = \sin x$?

A sine function can have an amplitude that is different from 1 as well as a period that is different from 2π. The equation of such a function is written in the form $y = a \sin bx$.

MATH TIP

The amplitude of $y = a \sin bx$, is $|a|$ and the period is $\frac{2\pi}{b}$.

9. Name the period and amplitude of $y = 8 \sin \frac{1}{4}x$.

10. Name the period and amplitude of $y = -\frac{1}{4}\sin 8x$.

Example D

Find the period and amplitude of $y = 3 \sin 6x$. Sketch its graph.

Step 1: The coefficient of sine, a, is 3. The amplitude is 3, because $|3| = 3$.

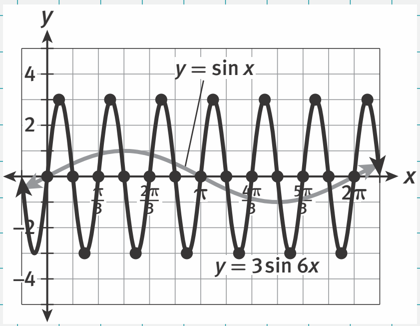
Step 2: The coefficient of x, b, is 6. Simplify $\frac{2\pi}{6}$ to show that the period is $\frac{\pi}{3}$. That means the graph completes one full cycle between 0 and $\frac{\pi}{3}$.

Step 3: Lightly sketch the parent sine function from 0 to 2π.

Step 4: Plot points at $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$, and 2π on the x-axis to show six complete cycles of $y = 3 \sin 6x$.

Step 5: Plot points on the x-axis halfway between those points to show where the curve crosses the x-axis when going between the maximum and minimum.

My Notes



Step 6: Plot maximum or minimum points, 3 and −3, halfway between each two zeros of the function.

Step 7: Connect the points with a smooth curve. Label the function.

Try These D

Name the period and amplitude of each function. Graph at least one period of each on a separate coordinate plane.

a. $y = -2\sin\frac{2}{3}x$

b. $y = 3\sin 3x$

c. $y = \frac{1}{4}\sin 4x$

d. $y = \frac{1}{2}\sin\frac{1}{2}x$

Check Your Understanding

11. How far must you extend the x- and y-axes to show one period of $y = 3\sin\frac{1}{3}x$?

12. Reason quantitatively. What is an equation for a sine function whose amplitude and period are both $\frac{3}{4}$ times the amplitude and period of $y = \sin\theta$? How did you find it?

13. Construct viable arguments. Suppose a classmate says that to stretch the parent sine function vertically so its amplitude is double, you would use the equation $y = 2\sin x$. To stretch it horizontally so one period is twice as wide, you would use the equation $y = \sin 2x$. Do you agree or disagree? Explain.

LESSON 34-2 PRACTICE

Name the period and amplitude of each function. Graph at least one period of each function on a separate coordinate plane.

14. $y = \frac{1}{2}\sin x$

15. $y = 2\sin x$

16. $y = \sin\frac{1}{2}x$

17. $y = \sin 2x$

18. $y = \frac{2}{3}\sin 4x$

19. $y = 4\sin\frac{2}{3}x$

20. Reason abstractly. How can you use the formula $period = \frac{2\pi}{b}$ to explain why the period decreases when $b > 1$ but increases when $0 < b < 1$?

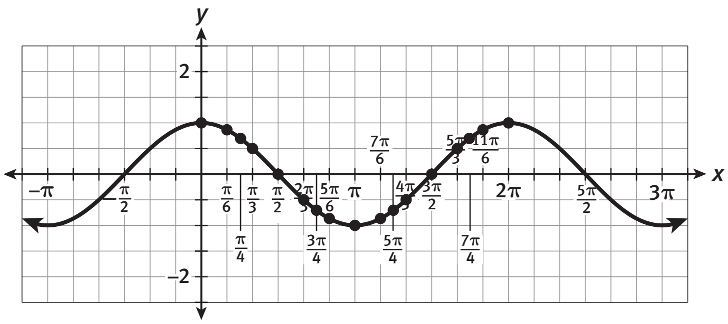
My Notes

Learning Targets:

- Graph the cosine function, $y = a \cos bx$.
- Find the period, midline, and amplitude of cosine functions.

SUGGESTED LEARNING STRATEGIES: Chunking the Activity, Close Reading, Paraphrasing, Create Representations, Discussion Groups, Think-Pair-Share

The cosine function is another example of a periodic function. It also repeats every 2π radians or 360°. You can use a table of trigonometric values to plot the values you know and then connect them to show the graph of the cosine function.



The graph of the cosine function is symmetrical with respect to the y-axis. Therefore, it is an *even function*. Another way to tell that it is an even function is to see that the values for $\cos x$ and $\cos(-x)$ are always the same.

MATH TIP

Recall that an *even function* is symmetric with respect to the y-axis. For each x, $f(-x) = f(x)$.

Check Your Understanding

1. Name the period, the midline, and the amplitude of the cosine function.
2. What is the maximum value of $\cos x$? List two x-values where the maximum occurs?
3. What is the minimum value of $\cos x$? List an x value where this minimum occurs.
4. Describe the similarities and differences between the maximums and minimums of the sine and cosine functions.

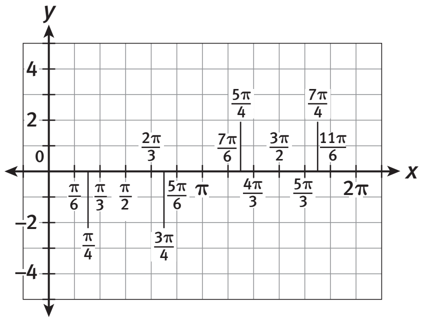
My Notes

TECHNOLOGY TIP

Make sure your calculator is set to use radians instead of degrees.

Changes to the equation of the parent cosine function, $y = \cos x$, will transform its graph. One possible change is for the cosine function to have a coefficient other than 1, written as $y = a \cos x$ where a is the coefficient of the function.

5. **Use appropriate tools strategically.** Use a graphing calculator to compare the graphs of $y = \cos x$, $y = 4 \cos x$, and $y = \frac{2}{3} \cos x$. Sketch and label the three graphs on the coordinate plane below.



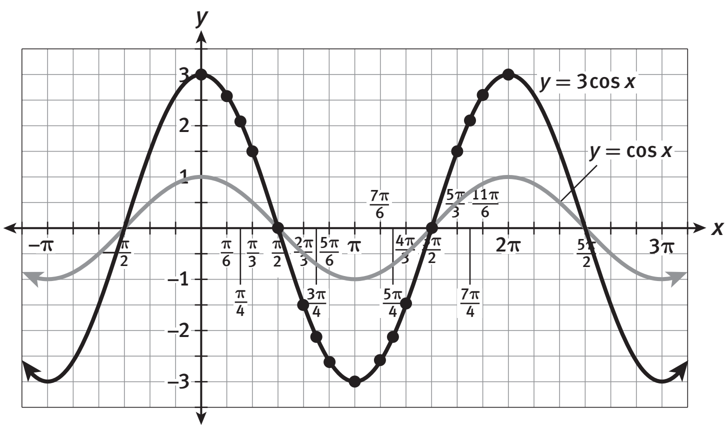
The periods, midlines, and x-intercepts are identical for all three graphs, but the amplitudes are different. The amplitude of $y = 4 \cos x$ is 4 and the graph is vertically stretched. The amplitude of $y = \frac{2}{3} \cos x$ is $\frac{2}{3}$ and the graph is vertically compressed. The amplitude of the function $y = a \cos x$ is $|a|$. As with the sine function, multiplying the cosine function by a coefficient changes the amplitude.

Example A

Draw the graph of $y = 3 \cos x$.
Name its period, amplitude, and midline.

Step 1: Lightly sketch the parent cosine function.

Step 2: Find several key points on the curve. Multiply the y value of each by 3 and plot the new point. Connect the points.



The period is 2π, the amplitude is 3, and the midline is $y = 0$.

My Notes

Try These A

Graph each cosine function on a separate coordinate plane. Name the period, amplitude, and midline for each.

a. $y = 5 \cos x$ **b.** $y = \frac{3}{4} \cos x$ **c.** $y = -2\cos x$.

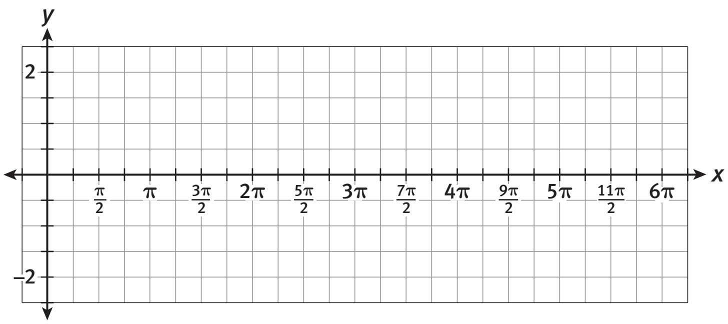
d. What is an equation of a cosine function that has a period of 2π, an amplitude of $\frac{5}{2}$, and a midline of $y = 0$?

The graph of the parent cosine function is also transformed when the angle has a coefficient other than 1, written as $y = \cos bx$ where b is the coefficient of the angle.

6. Use a graphing calculator to compare the graphs of $y = \cos x$, $y = \cos 5x$, and $y = \cos \frac{1}{4}x$. Sketch and label the 3 graphs on the coordinate plane below.

TECHNOLOGY TIP

Remember to set the WINDOW of your graphing period to show at least one full period of the function.



Notice that the amplitudes and midlines are identical for all three graphs. The periods are different. The graph of $y = \cos 5x$ is horizontally compressed so that it shows five cycles between 0 and 2π. It repeats every $\frac{2\pi}{5}$ units so its period is $\frac{2\pi}{5}$. The graph of $y = \cos \frac{1}{4}x$ is horizontally stretched so that it shows one-fourth of a cycle between 0 and 2π. It shows one full cycle between 0 and 8π. Its period is 8π.

The period of the function $y = \cos bx$ is $\frac{2\pi}{b}$. It is found by dividing 2π by the coefficient of the angle.

Example B

Find the period of $y = \cos \frac{1}{3}x$. Sketch its graph.

Step 1: The coefficient of x is $\frac{1}{3}$. Simplify $\frac{2\pi}{\frac{1}{3}}$ to find that the period is 6π. The graph completes one full cycle between 0 and 6π.

Step 2: Lightly sketch the parent cosine function from 0 to 6π.

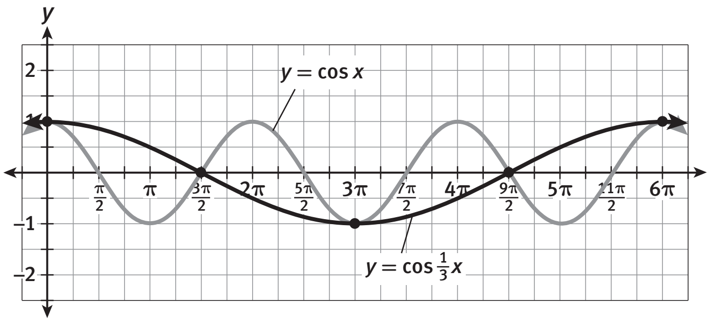
My Notes

Step 3: A cycle of the cosine function begins and ends with its maximum. Plot points at (0, 1) and $(6\pi, 1)$ to show the beginning and end of one cycle of $y = \cos \frac{1}{3}x$.

Step 4: Plot a point on $(3\pi, -1)$. It is the halfway point of the cycle. The minimum of the cosine is halfway through the cycle.

Step 5: The zeros of $y = \cos \frac{1}{3}x$ are halfway between the maximums and minimums of the function. Plot the points $\left(\frac{3\pi}{2}, 0\right)$ and $\left(\frac{9\pi}{2}, 0\right)$.

Step 6: Connect the points with a smooth curve. Label the function.



Example C

Find the period of $y = \cos 4x$. Sketch its graph.

Step 1: The coefficient of x is 4. Simplify $\frac{2\pi}{4}$ to find that the period is $\frac{\pi}{2}$. The graph completes one full cycle between 0 and $\frac{\pi}{2}$.

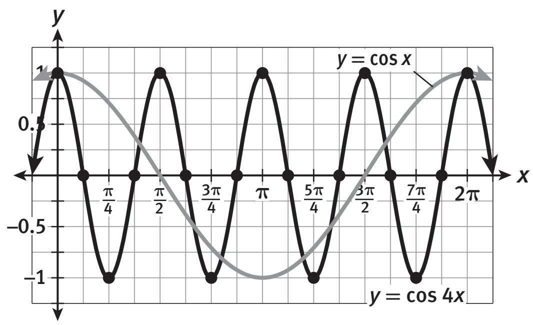
Step 2: Lightly sketch the parent cosine function.

Step 3: Because $\cos 0 = 1$, the beginning of each cycle will be a maximum. Plot points at (0, 1), $\left(\frac{\pi}{2}, 1\right)$, $(\pi, 1)$, $\left(\frac{3\pi}{2}, 1\right)$, and $(2\pi, 1)$ to show four complete cycles of $y = \cos 4x$.

Step 4: The minimums are halfway between the maximums. Show these with points at $\left(\frac{\pi}{4}, -1\right)$, $\left(\frac{3\pi}{4}, -1\right)$, $\left(\frac{5\pi}{4}, -1\right)$, and $\left(\frac{7\pi}{4}, -1\right)$.

Step 5: The zeros occur between each maximum and minimum. Locate them at $\left(\frac{\pi}{8}, 0\right)$, $\left(\frac{3\pi}{8}, 0\right)$, $\left(\frac{5\pi}{8}, 0\right)$, $\left(\frac{7\pi}{8}, 0\right)$, $\left(\frac{9\pi}{8}, 0\right)$, $\left(\frac{11\pi}{8}, 0\right)$, $\left(\frac{13}{8}, 0\right)$, and $\left(\frac{15\pi}{8}, 0\right)$.

Step 6: Connect the points with a smooth curve. Label the function.



Try These B–C

Find the period of the functions. Graph at least one period of each on a separate coordinate plane. Then name the amplitude and midline of each graph.

a. $y = \cos 2x$

b. $y = \cos \frac{1}{2}x$

c. What is an equation of a cosine function that has a period of 3π, an amplitude of 1, and a midline of $y = 0$?

My Notes

Check Your Understanding

7. Model with mathematics. How is the graph of $y = \frac{3}{4}\cos 2x$ similar to the graph of $y = \frac{3}{4}\sin 2x$.

8. How is the graph of $y = \frac{3}{4}\cos 2x$ different from the graph of $y = \frac{3}{4}\sin 2x$.

Cosine functions can have both amplitudes and periods that are different from those of the parent function $y = \cos x$. Changes in the amplitude and period can be shown by the equation $y = a \cos bx$.

MATH TIP

The amplitude of $y = a\cos bx$, is $|a|$ and the period is $\frac{2\pi}{b}$.

9. Name the period and amplitude of $y = 6\cos \frac{1}{3}x$.

10. Name the period and amplitude of $y = -\frac{1}{3}\cos 6x$.

Example D

Find the period and amplitude of $y = 2 \cos 3x$. Sketch its graph.

Step 1: The coefficient of cosine, a, is 2. The amplitude is 2 because $|2| = 2$.

Step 2: The coefficient of x, b, is 3. The graph will have 3 cycles of $2 \cos 3x$ between 0 and 2π. The period is $\frac{2\pi}{3}$ which means the graph completes one full cycle between 0 and $\frac{2\pi}{3}$.

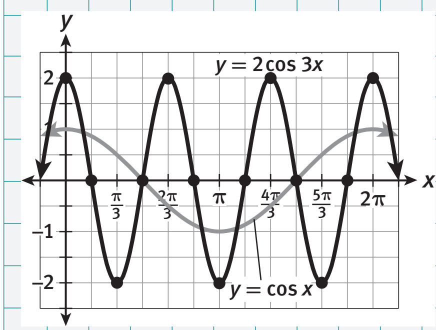
Step 3: Lightly sketch the parent cosine function from 0 to 2π.

Step 4: The maximums, 2, will occur at $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$, and 2π. Locate them on the graph.

Step 5: The minimum, -2, will occur halfway between the maximums. Locate them on the graph.

Step 6: The zeros will fall between each maximum and minimum. Locate them on the graph.

Step 7: Connect the points with a smooth curve. Label the function.



My Notes

Try These D

Name the period and amplitude of each function. Graph at least one period of each on a separate coordinate plane.

a. $y = 4\cos 2x$

b. $y = -5\cos\frac{1}{2}x$

c. $y = \frac{2}{3}\cos\frac{2}{3}x$

d. $y = 4\cos 4x$

Check Your Understanding

11. How does the parent cosine function change when it is stretched vertically? Stretched horizontally?

12. How does the parent cosine function change when it is compressed vertically? Compressed horizontally?

13. **Use appropriate tools strategically.** For what value(s) of x does $\sin x = \cos x$ over the interval $0 \le x \le 2\pi$? Does $\sin 2x = \cos 2x$ for the same values of x as $\sin x = \cos x$? Explain.

TECHNOLOGY TIP

You can use a graphing calculator to see where two functions are equal. Graph them both and see where they intersect. You can use the TABLE or TRACE keys to get a fairly exact point of intersection.

LESSON 34-3 PRACTICE

Write your answers on notebook or graph paper.

Name the period and amplitude of each function. Graph at least one period of each on a separate coordinate plane.

14. $y = \frac{1}{2}\cos x$

15. $y = 2\cos x$

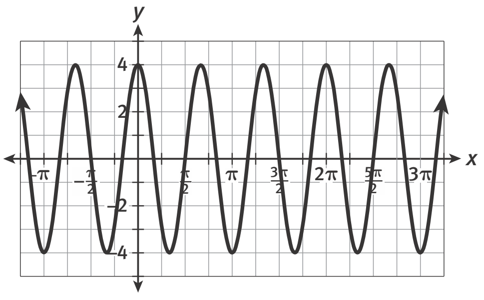
16. $y = \cos\frac{1}{2}x$

17. $y = \cos 2x$

18. $y = \frac{2}{3}\cos 4x$

19. $y = 4\cos\frac{2}{3}x$

20. **Make sense of problems.** What is the equation of this graph in the form $y = a\cos bx$? How did you determine the values of a and b?



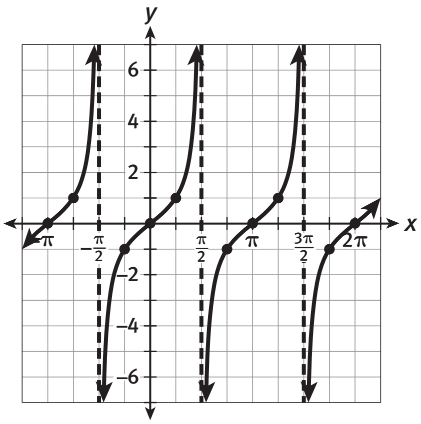
My Notes

Learning Targets:

- Graph the tangent function, $y = a \tan b x$.
- Find the period, and midline of tangent functions.

SUGGESTED LEARNING STRATEGIES: Chunking the Activity, Close Reading, Paraphrasing, Create Representations, Discussion Groups, Think-Pair-Share

The tangent function is another periodic function. You can use a table of trigonometric values to plot the values you know and then connect them to show the graph of the tangent function.



1. What is the shape of the graph?

2. What is the horizontal distance required for the graph of the tangent function to complete one cycle?

3. What is the range of the graph?

4. Are there any gaps in the graph? Where?

5. Is tangent an even or odd function? Explain.

To summarize, the tangent function is an odd periodic function with a period of π. It extends from negative to positive infinity so we do not refer to its amplitude. The curve approaches $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, and $x = \frac{3\pi}{2}$ but never intersects them and $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, and $x = \frac{3\pi}{2}$ are *asymptotes* of $y = \tan x$. Because the tangent function is undefined at these points, the domain of $\tan x$ is $x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ or any odd multiple of $\frac{\pi}{2}$. The midline is $y = 0$. The zeros are at multiples of π.

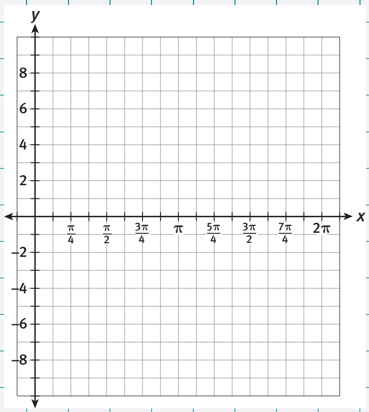
MATH TIP

Recall that an *asymptote* is a line that a graph approaches but does not intersect.

My Notes

MATH TIP

$\tan x = \frac{\sin x}{\cos x}$



Check Your Understanding

6. **Make use of structure.** What is $\tan x$ when $\sin x = 0$? Why?
7. What is $\tan x$ when $\cos x = 0$? Why? How is this shown on the graph?
8. What is $\tan x$ when $\sin x = \cos x$? Why? At what values of x does this occur?

Multiplying the parent function $y = \tan x$ by a coefficient, $y = a \tan x$, transforms the graph of the function.

9. Use a graphing calculator to compare the graphs of $y = \tan x$, $y = 3 \tan x$, and $y = \frac{1}{3} \tan x$. Sketch and label the 3 graphs on the coordinate plane below.

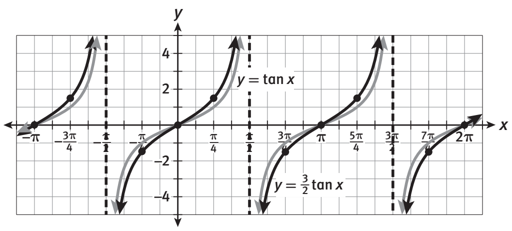
The periods, midlines, and x-intercepts are identical for all three graphs, but the shapes are different. The graph of $y = 3 \tan x$ is narrower than $y = \tan x$ and approaches the asymptotes more slowly – it has been vertically stretched. The graph of $y = \frac{1}{3} \tan x$ is wider and approaches the asymptotes more quickly—it has been vertically compressed.

Example A

Draw the graph of $y = \frac{3}{2} \tan x$.
Name its period, midline, and asymptotes.

Step 1: Lightly sketch the parent tangent function.

Step 2: Find several key points on the curve. Multiply the y value of each by 3 and plot the new point. Connect the points.



The period is π, the midline is $y = 0$ and the asymptotes are $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, and $x = \frac{3\pi}{2}$.

Try These A

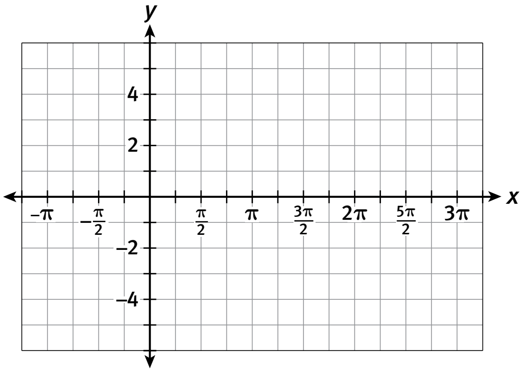
Graph each tangent function on a separate coordinate plane. State the period and state whether each function is vertically stretched or compressed relative to the parent function.

a. $y = 2 \tan x$ **b.** $y = \frac{5}{2} \tan x$ **c.** $y = \frac{1}{2} \tan x$.

My Notes

The graph of the parent tangent function is also transformed when the coefficient of x is a value other than 1, written as $y = \tan bx$. Changing the coefficient of x changes the period of the function.

10. Use a graphing calculator to compare the graphs of $y = \tan x$, $y = \tan 3x$, and $y = \tan \frac{1}{3}x$. Sketch and label the 3 graphs on the coordinate plane below.



Notice that the midlines are identical for all three graphs. The periods and asymptotes are different. The graph of $y = \tan 3x$ completes 3 cycles in the same interval it takes $y = \tan x$ to complete 1 cycle. Its period is $\frac{\pi}{3}$, and it is horizontally compressed. The graph of $y = \tan \frac{1}{3}x$ completes one cycle in the same interval that $y = \tan x$ completes three cycles. Its period is 3π, and it is horizontally stretched.

The period of the function $y = \tan b\theta$ is $\frac{\pi}{b}$. It is found by dividing π by the coefficient of the angle.

The asymptotes of the functions are also different. Some asymptotes of $y = \tan x$ are $x = \frac{\pi}{2}$ and odd multiples of $\frac{\pi}{2}$. The asymptotes of $y = \tan 3x$ are closer together and are found at $x = \frac{\pi}{6}$ and odd multiples of $\frac{\pi}{6}$. The asymptotes of $y = \frac{1}{3}x$ are farther apart and are found at $x = \frac{3\pi}{2}$ and odd multiples of $\frac{3\pi}{2}$.

Discuss Items 11–14 with your group.

11. Construct viable arguments. How is the formula for finding the period of the tangent function different from the formulas for finding the periods of the sine and cosine functions? Why do you think this is?

DISCUSSION GROUP TIP

As you share your ideas, be sure to use mathematical terms and academic vocabulary precisely. Make notes to help you remember the meaning of new words and how they are used to describe mathematical concepts.

12. Make use of structure. Compare the periods and asymptotes of $y = \tan x$ and $y = \tan 3x$. What pattern do you see?

My Notes

13. Compare the periods and asymptotes of $y = \tan$ and $y = \tan \frac{1}{3}x$ as well as their asymptotes. What pattern do you see?

14. Reason abstractly. Use your answers to Items 7 and 8 to predict the asymptotes of $y = \tan \frac{3}{2}x$. Explain your thinking.

Example B

Find the period of $y = \tan \frac{3}{2}x$. Sketch its graph. Name its asymptotes.

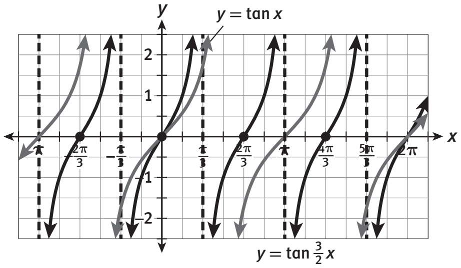
Step 1: The coefficient of x is $\frac{3}{2}$. Simplify $\frac{\pi}{\frac{3}{2}}$ to find that the period is $\frac{2}{3}\pi$. That means the horizontal distance of one period of the graph is $\frac{2}{3}\pi$.

Step 2: Lightly sketch the parent tangent function from $-\pi$ to 2π.

Step 3: Find zeros of $y = \tan \frac{3}{2}x$ by dividing zeros of $y = \tan x$ by $\frac{3}{2}$. Some zeros of $y = \tan \frac{3}{2}x$ are $-\frac{2\pi}{3}$, 0, $\frac{2\pi}{3}$, and $\frac{4\pi}{3}$.

Step 4: Find the asymptotes of $y = \tan \frac{3}{2}x$ by dividing asymptotes of $y = \tan x$ by $\frac{3}{2}$. Some asymptotes of $y = \tan \frac{3}{2}x$ are $y = -\pi$, $y = -\frac{\pi}{3}$, $y = \frac{\pi}{3}$, $y = \pi$, and $y = \frac{5\pi}{3}$. Sketch the asymptotes. Notice that the asymptotes are halfway between each pair of zeros.

Step 5: Draw the curves for $y = \tan \frac{3}{2}x$, having them cross the midline at the zeros and approach, but not touch, the asymptotes.



Try These B

Find the period, some zeros, and some asymptotes of the functions. Graph at least one period of each on a separate coordinate plane.

a. $y = \tan 2x$

My Notes

b. $y = \tan \frac{1}{2}x$

c. What is an equation of a tangent function with a period of $\frac{5}{2}\pi$?

d. What is an equation of a tangent function with a period of $\frac{2}{5}\pi$?

Check Your Understanding

15. How could you use the function $y = \sin 2x$ to find the zeros of $y = \tan 2x$?

16. How could you use the function $y = \cos 2x$ to find the asymptotes of $y = \tan 2x$?

When you graph $y = a \tan bx$, the value of a compresses or stretches the graph vertically. The value of b compresses or stretches it horizontally.

MATH TIP

The period of $y = a \tan bx$ is $\frac{\pi}{b}$. When $0 < a < 1$, the graph approaches the asymptotes more quickly than the graph of $y = \tan x$. When $a > 1$, the graph approaches the asymptotes more slowly than the graph of $y = \tan x$.

Example C

Find the period and asymptotes of $y = \frac{1}{2}\tan \frac{1}{2}x$. Sketch its graph.

Step 1: Lightly sketch the parent tangent function.

Step 2: The coefficient of the angle, b, is $\frac{1}{2}$. The period is $\frac{\pi}{\frac{1}{2}}$ or 2π. The horizontal width of one period is 2π.

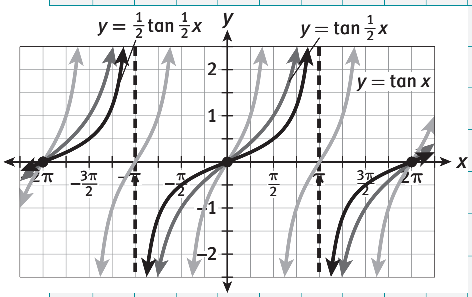
Step 3: Divide zeros of $y = \tan x$, $-\pi$, 0, and π, by $\frac{1}{2}$ to find the zeros of $y = \frac{1}{2}\tan \frac{1}{2}x$. They are -2π, 0, and 2π. Locate them on the graph.

Step 4: Divide asymptotes of $y = \tan x$, $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$, by $\frac{1}{2}$ to find the asymptotes of $y = \frac{1}{2}\tan \frac{1}{2}x$. They are $x = -\pi$ and $x = \pi$. Sketch them on the graph.

Step 5: Lightly sketch $y = \tan \frac{1}{2}x$ from -2π to 2π.

Step 6: Note that the value of a on $y = \frac{1}{2}\tan \frac{1}{2}x$ is $\frac{1}{2}$. Locate some points on each portion of the curve of $y = \tan \frac{1}{2}x$. Multiply the y-values by $\frac{1}{2}$. Plot the points with the new y-values.

Step 7: Connect the points and zeros with a smooth curve. Do not let it intersect the asymptotes. Label the function.



My Notes

MATH TIP

The asymptotes of a tangent function are located halfway between the zeros of the function.

Try These C

Name the period, zeros, and asymptotes of each function. Graph at least one period of each on a separate coordinate plane.

a. $y = \frac{2}{3}\tan\frac{3}{2}x$

b. $y = \frac{3}{2}\tan\frac{2}{3}x$

Check Your Understanding

17. Write an equation of the tangent function where the curves of the tangent function will be closer together than in $y = \tan x$.

18. Write an equation of the tangent function where the curves of the tangent function approach the asymptotes more quickly than in $y = \tan x$.

LESSON 34-4 PRACTICE

Model with mathematics. Name the period, zeros, and asymptotes of each function. Graph at least one period of each on a separate coordinate plane.

19. $y = \frac{5}{4}\tan x$

20. $y = \frac{4}{5}\tan x$

21. $y = \tan 4x$

22. $y = \tan\frac{1}{3}x$

23. $y = \frac{4}{3}\tan\frac{2}{3}x$

24. $y = \frac{2}{3}\tan\frac{4}{3}x$

25. **Critique the reasoning of others.** Dianne notices that the graphs of $y = 2\tan x$ and $y = \tan 2x$ are both narrower than the graph of $y = \tan x$. She concludes that they are equivalent graphs. Do you agree? Explain.

My Notes

Learning Targets:

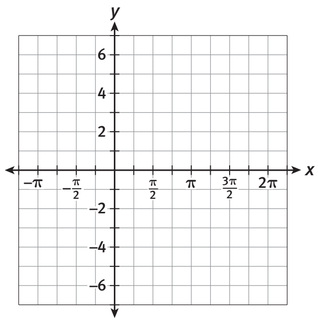
- Describe and graph functions of the form $y = a \sin b(x - h) + k$, $y = a \cos b(x - h) + k$, and $y = a \tan b(x - h) + k$.
- Find the period, amplitude, and midline of these trigonometric functions.

SUGGESTED LEARNING STRATEGIES: Close Reading, Paraphrasing, Create Representations, Discussion Groups, Identify a Subtask, Think-Pair-Share

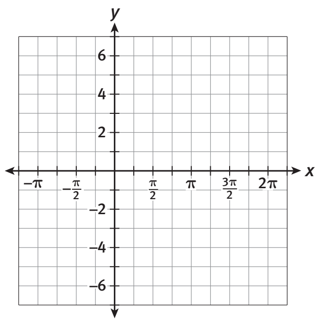
The transformations in the previous lesson stretched or compressed the graphs of trigonometric functions. It is also possible to translate the graphs of trigonometric functions without changing their shapes.

1. Use appropriate tools strategically. For each item, use a graphing calculator to compare the three graphs. Sketch and label them on the coordinate plane. Then compare and contrast the graphs of the three functions

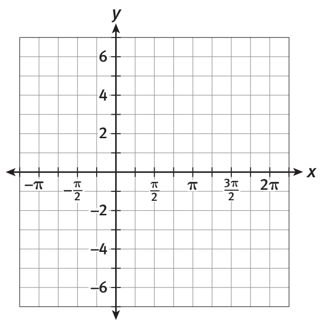
a. $y = \sin x$, $y = \sin x + 3$, and $y = \sin x - 2$.



b. $y = \cos x$, $y = \cos x + 4$, and $y = \cos x - 1$



c. $y = \tan x$, $y = \tan x + 3$, and $y = \tan x - 2$



TECHNOLOGY TIP

Remember to set the WINDOW of your graphing period to show at least one full period of the function.

My Notes

MATH TIP

When $k > 0$, the graph of $y = \sin x + k$ shifts k units up, and the graph of $y = \sin x - k$ shifts k units down.

The midline is the only feature that changes in each set of graphs. The amplitudes and periods stay the same. Adding or subtracting a constant causes the graph of the parent function to translate up or down. When you graph $y = \sin + k$, $y = \cos x + k$, and $y = \tan x + k$ the graph is translated k units vertically. The midline is also vertically translated by k units.

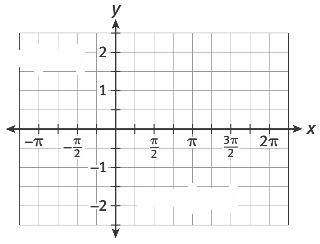
Check Your Understanding

2. Are the graphs of $y = 2\cos x$ and $y = \cos x + 2$ the same? Explain.

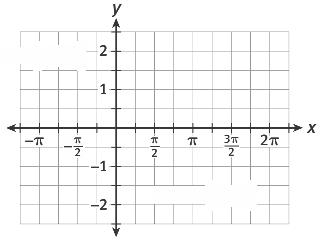
It is also possible to translate trigonometric functions horizontally.

3. Use appropriate tools strategically. For each item, use a graphing calculator to compare the three graphs. Sketch and label them on the coordinate plane. Then compare and contrast the graphs of the three functions

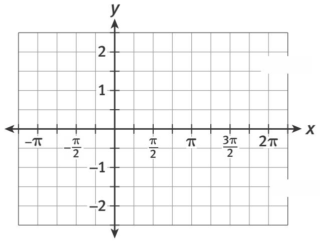
a. $y = \sin x$, $y = \sin\left(x + \frac{\pi}{4}\right)$, and $y = \sin\left(x - \frac{\pi}{2}\right)$.



b. $y = \cos x$, $y = \cos\left(x + \frac{\pi}{3}\right)$, and $y = \cos\left(x - \frac{2\pi}{3}\right)$.



c. $y = \tan x$, $y = \tan\left(x + \frac{\pi}{2}\right)$, and $y = \tan\left(x - \frac{\pi}{4}\right)$.



MATH TIP

When $h > 0$, the graph of $y = \sin(x - h)$ shifts h units to the right, and the graph of $y = \sin(x + h)$ shifts h units to the left.

My Notes

The graphs in each set are the same except for their horizontal positions. Adding or subtracting a constant to the angle causes the graph of the parent function to translate left or right. When you graph $y = \sin(x - h)$, $y = \cos(x - h)$, and $y = \tan(x - h)$ the graph is translated h units horizontally. Another name for a horizontal shift of a periodic function is a ***phase shift***. The zeros also slide horizontally the same distance, as do the asymptotes of the tangent function.

MATH TERMS

The **phase shift** of a periodic function is the distance it is translated horizontally from its parent function.

Example A

Describe the vertical and horizontal shifts of $y = \cos(x - \pi) + 1$. Sketch its graph. Name its midline, at least one maximums, and at least one minimum.

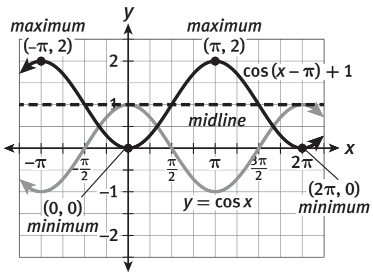
Step 1: The vertical shift is 1 unit up. The horizontal shift is π units to the right.

Step 2: Lightly sketch the parent cosine function from $-\pi$ to 2π.

Step 3: Sketch and label the graph of $y = \cos(x - \pi) + 1$ one unit above and π units to the right of the graph of $y = \cos x$.

Step 4: Locate and name one maximum and one minimum.

Step 5: Sketch the horizontal line $y = 1$ which is halfway between the minimums and maximums. This is the midline.



Example B

Describe the vertical and horizontal shifts of $y = \tan\left(x + \frac{\pi}{4}\right) - 2$. Sketch its graph. Name its midline and at least one asymptote.

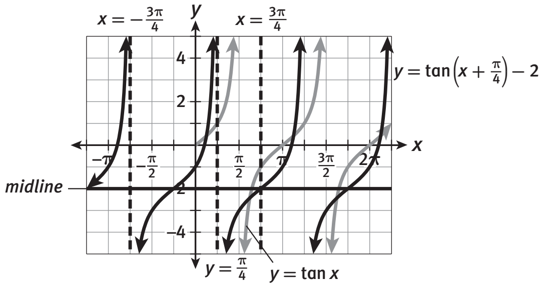
Step 1: The vertical shift is 2 units down. The horizontal shift is $\frac{\pi}{4}$ units to the left.

Step 2: Lightly sketch the parent tangent function from 0 to 2π.

Step 3: Sketch and label the graph of $y = \tan\left(x + \frac{\pi}{4}\right) - 2$ two units below and $\frac{\pi}{4}$ units to the left of the graph of $y = \tan x$.

Step 4: Sketch the horizontal line $y = -2$. This is the midline.

Step 5: Sketch the asymptotes.



My Notes

Try These A–B

Describe the vertical and horizontal shifts of each function.
Graph at least one period of each on a separate coordinate plane. Draw and label the midline and any asymptotes.

a. $y = \sin\left(x + \frac{\pi}{3}\right) + 2$ **b.** $y = \tan\left(x - \frac{\pi}{2}\right) - 1$

The chart summarizes the transformations of the parent sine, cosine, and tangent functions that you have learned in this activity.

$y = a \sin b(x - h) + k$ $y = a \cos b(x - h) + k$ $y = a \tan b(x - h) + k$	
a	• The coefficient changes the amplitude of the sine and cosine functions. • When $a > 1$, the amplitude increases and the graph is stretched vertically. • When $0 < a < 1$, the amplitude decreases and the graph is compressed vertically. • When $a < 0$, the graph is reflected across the x-axis.
b	• The coefficient changes the period. • When $b > 1$, the period decreases and the graph is compressed horizontally. • When $0 < b < 1$, the period increases and the graph is stretched horizontally. • The period of $\sin bx$ and $\cos bx$ is $\frac{2\pi}{b}$. • The period of $\tan bx$ is $\frac{\pi}{b}$.
h	• The constant shifts the graph horizontally. • When $h > 0$, the graph shifts to the right. • When $h < 0$, the graph shifts to the left.
k	• The constant shifts the graph vertically. • When $k > 0$, the graph shifts up. • When $k < 0$, the graph shifts down.

Example C

List the features of $y = 2\sin 2\left(x + \frac{\pi}{4}\right) + 3$. Sketch its graph.

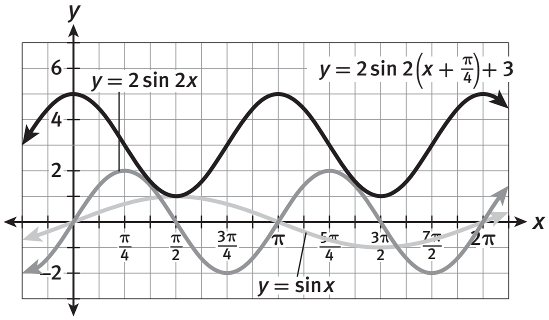
Step 1: The amplitude is 2, the period is π, horizontal shift is $\frac{\pi}{4}$ to the left, and the vertical shift is 3 up.

Step 2: Lightly sketch the parent sine function.

My Notes

Step 3: Lightly sketch the graph of $y = 2\sin 2x$. The amplitude is doubled by a factor of 2 and the period is compressed by a factor of $\frac{1}{2}$.

Step 4: Sketch that curve again, shifting it $\frac{\pi}{4}$ units to the left and 3 units up to show the graph of $y = 2\sin 2\left(x + \frac{\pi}{4}\right) + 3$. Label it.



Try These C

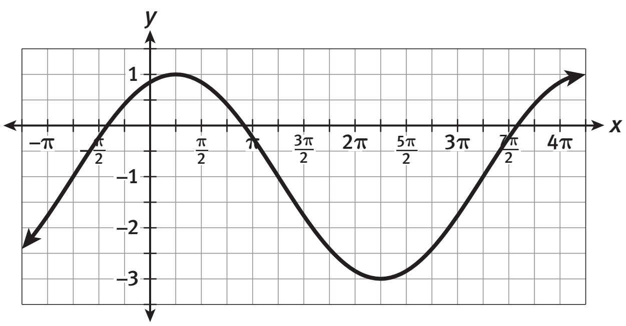
List the features of each function. Sketch its graph.

a. $y = \frac{4}{3}\cos\frac{1}{2}\left(x - \frac{\pi}{3}\right) + 2$ **b.** $y = \frac{3}{4}\tan 2\left(x + \frac{\pi}{3}\right) - 1$

c. What is an equation of a sine function with amplitude of 2, a period of π, a horizontal shift of $\frac{2\pi}{3}$ units to the right, and a vertical shift of 3 units down?

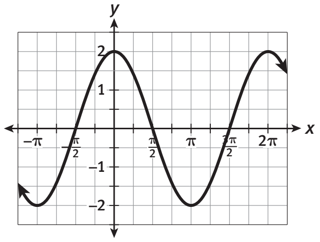
Check Your Understanding

4. Make sense of problems. Find the amplitude, period, horizontal shift, and vertical shift of the graph below. Write its equation.



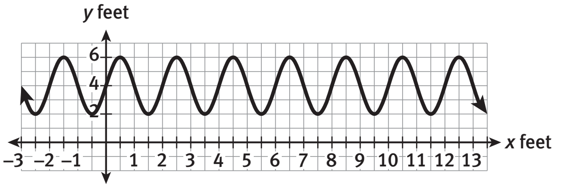
My Notes

5. **Construct viable arguments.** When asked to write the equation of the graph shown below, Samara says it is $y = 2\cos x$. Eduardo says it is $y = 2\sin\left(x + \frac{\pi}{2}\right)$. Which student is correct? Why?



6. Dominic used a graphing calculator to compare the graphs of $y = \tan x - 2$ and $y = \tan(x - \pi) - 2$. When he pressed GRAPH, the calculator displayed only one graph. What happened?

Look at the graph of the mural for the transit station that was presented at the beginning of the activity.



It is possible to write an equation that describes it accurately. Because it crosses the midline at $x = 0$, it would be easiest to use the sine function and write it in the form $y = a \sin b(x - h) + k$. Use information from the graph to find a, b, h, and k.

- The midline is $y = 4$. The function extends 2 feet above and 2 feet below the midline so the amplitude is 2.
- The graph completes one full cycle in 2 feet, so the period is 2. Solve $\frac{2\pi}{b} = 2$ for b.
- The graph has not been shifted horizontally, so the phase shift is 0. Because $h = 0$, it is not needed in the equation.
- The midline is $y = 4$ so the graph has been shifted 4 units up.

The equation for the design at the transit center is ______.

LESSON 34-5 PRACTICE

List the features of each function. Sketch its graph.

7. $y = \cos\frac{1}{2}(x - \pi) - 1$

8. $y = \frac{1}{3}\tan\left(x + \frac{2\pi}{3}\right) + 2$

9. $y = \frac{3}{2}\sin 2\left(x + \frac{\pi}{4}\right) + 1$

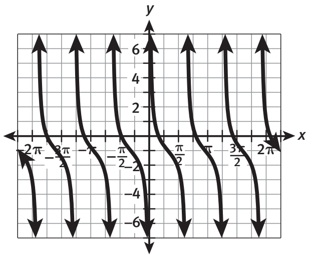
10. **Reason abstractly.** Why does adding a constant greater than zero to a trigonometric function move the graph up, but multiplying it by a coefficient greater than one stretches it?

ACTIVITY 34 PRACTICE

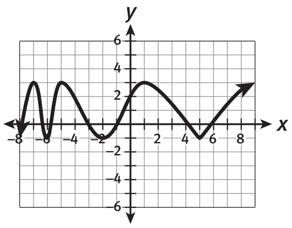
Lesson 34-1

State whether each graph in Items 1–4 shows a periodic function. If periodic, give the period, amplitude, and the equation of the midline. If not periodic, explain why not.

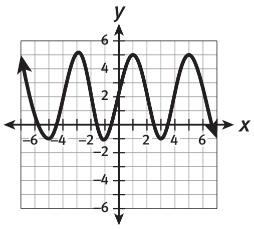
1.



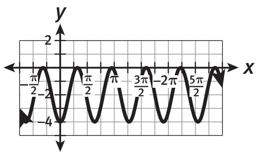
2.



3.



4.



5. How can you use the maximum and minimum y-values of a periodic function to find the equation of the midline?

6. Draw the graph of a periodic function that has a period of 3, an amplitude of 2.5, and a midline of $y = 0.5$.

Lesson 34-2

Name the period and amplitude of each function. Graph at least one period of each on a separate coordinate plane.

7. $y = 4 \sin x$

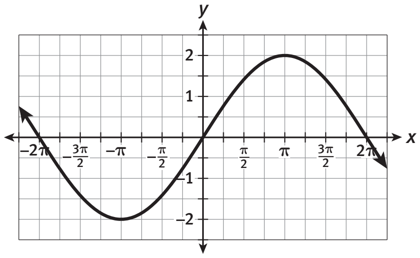
8. $y = \frac{1}{4} \sin x$

9. $y = \sin 4x$

10. $y = \sin \frac{1}{4} x$

11. $y = \frac{5}{2} \sin \frac{2}{5} x$

Refer to the graph below for Items 12–14.



12. What is the period and amplitude of the graph?

13. What is the equation of the function?

14. What is the equation of a graph that is half as wide and twice as tall as the one shown?

Lesson 34-3

Name the period and amplitude of each function. Graph at least one period of each on a separate coordinate plane.

15. $y = 3 \cos x$

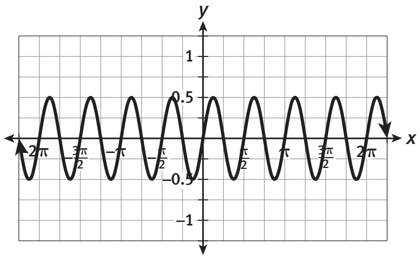
16. $y = \frac{2}{3} \cos x$

17. $y = \cos 3x$

18. $y = \cos \frac{2}{3} x$

19. $y = \frac{3}{2} \cos \frac{1}{3} x$

Refer to the graph below for Items 20–21.



20. What is the period and amplitude of the graph?

21. What is the equation of the function?

Suppose a graphic designer wanted to use the cosine function to create a mural. However, she wanted it to appear three times narrower than the parent cosine function. She was not sure whether to use the graph of $y = 3 \cos x$ or the graph of $y = \cos 3x$.

22. Graph $y = \cos x$ and $y = 3 \cos x$ on the same coordinate axis. Compare and contrast the graphs of the two functions.

23. Graph $y = \cos x$ and $y = \cos 3x$ on the same coordinate axis. Compare and contrast the graphs of the two functions

24. Which equation results in a graph three times narrower than $y = \cos x$? Explain.

Lesson 34-4

Name the period, zeros, and asymptotes of each function. Graph at least one period of each on a separate coordinate plane.

25. $y = \frac{3}{2}\tan x$

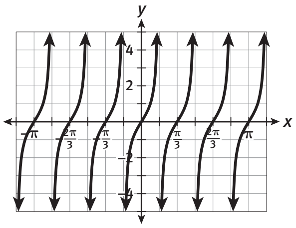
26. $y = \frac{1}{2}\tan x$

27. $y = \tan\frac{2}{3}x$

28. $y = \tan\frac{3}{2}x$

29. $y = 2\tan\frac{1}{4}x$

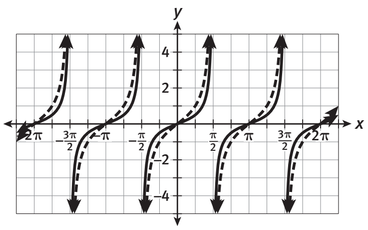
Refer to the graph below for Items 30–31.



30. Name the period, zeros, and asymptotes of the graph.

31. What is the equation of the function?

Refer to the graph below for Items 32–34.



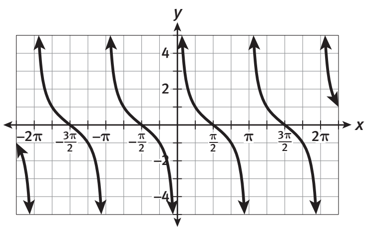
32. Name the period, zeros, and asymptotes of both graphs.

33. What is the value of $f\left(\frac{\pi}{4}\right)$ for the function shown by the dashed line?

34. What is the value of $f\left(\frac{\pi}{4}\right)$ for the function shown by the solid line?

35. What is the equation of the function shown by the dashed line? the solid line?

36. George graphed $y = \tan x$ as shown below. The teacher marked it wrong. He argued that the zeros and asymptotes were correct. He did not understand what was wrong with it. Explain why it is incorrect.



Lesson 34-5

For each function, describe the phase (horizontal) shift and vertical shift relative to the parent function. Then graph it.

37. $y = \cos\left(x + \frac{\pi}{4}\right) + 2$

38. $y = \cos\left(x - \frac{2\pi}{3}\right) - 3$

39. $y = \tan\left(x - \frac{\pi}{3}\right) + 2$

40. $y = \tan\left(x + \frac{\pi}{2}\right) - 1$

41. $y = \sin\left(x - \frac{\pi}{2}\right) - 2$

42. $y = \sin(x + \pi) + 1$

Describe the meaning of the "2" in each function and its effect on the graph of each function relative to the parent function.

43. $y = 2\cos x$

44. $y = \cos 2x$

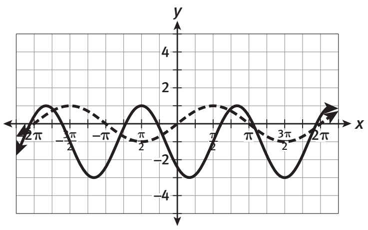
45. $y = \cos(x + 2)$

46. $y = \cos x + 2$

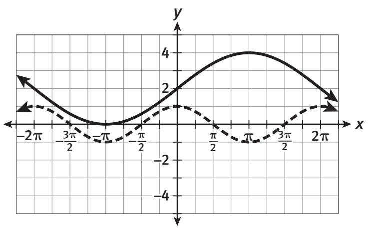
MATHEMATICAL PRACTICES
Make Sense of Problems

In each graph below, the parent trigonometric function is shown with a dashed line. Name the amplitude change (a), period change (b), phase (horizontal) shift (h), and vertical shift (k) shown by the function graphed with a solid line. Then write its equation in the form $y = a\sin b(x - h) + k$, $y = a\cos b(x - h) + k$, or $y = a\tan b(x - h) + k$.

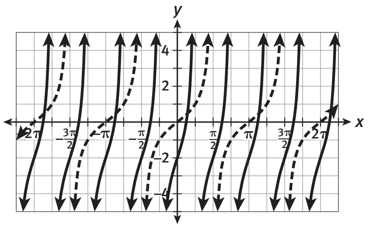
47.



48.



49.



Choosing Functions to Model Periodic Phenomena

The Sky Wheel

Lesson 35-1 Modeling Periodic Phenomena

My Notes

Learning Targets:

- Use trigonometric functions to model real-world periodic phenomena.
- Identify key features of these functions.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Visualization, Look for a Pattern, Think-Pair-Share, Group Presentation

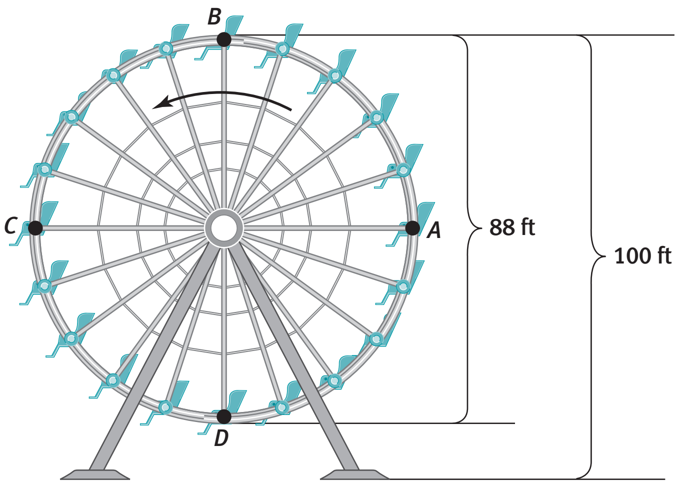
Tyrell is an engineer at Rocket Rides. His job is to design amusement park rides that meet the needs of the company's clients. A new client has asked Rocket Rides to design a Ferris wheel, given the following ***constraints***.

- The diameter of the wheel must be 88 feet.
- The highest point of the wheel must be 100 feet above ground.
- The wheel must make one rotation every 60 seconds.

Based on this information, Tyrell creates a preliminary sketch for a ride called The Sky Wheel, as shown.

ACADEMIC VOCABULARY

A ***constraint*** is a condition or restriction that must be satisfied. For example, biologists might study the physical constraints that determine the possible sizes and shapes of insects.



Tyrell wants to write a function that models the motion of the Ferris wheel. He starts by considering the motion of a car that begins at point *A*.

1. **Reason quantitatively.** What is the height of the car when it is at point *A*? Explain?

My Notes

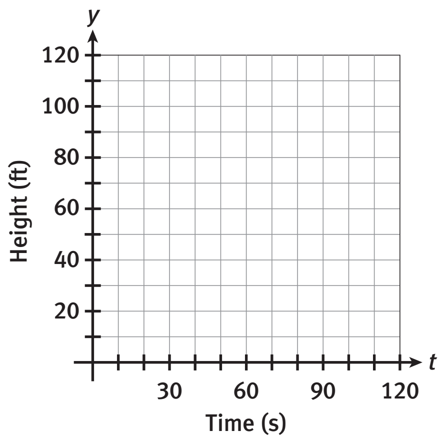
2. Tyrell makes a table showing the height of the car in feet at various times as the Ferris wheel rotates counterclockwise. Complete the table.

Time (seconds)	0	15	30	45	60	75	90	105	120
Height (feet)									

3. Describe any patterns you see in the table.

4. Consider the height of the car as a function of time. Is the function periodic? If so, what is the period? Explain.

5. **Model with mathematics.** Plot the points from the table in Item 2 and connect them with a smooth curve. (You will get a chance to determine the precise shape of the curve later.)



6. What is the amplitude of the function you graphed?

7. What is the equation of the midline?

My Notes

8. Tyrell plans to write an equation for the function in the form $f(t) = a \sin b(t - h) + k$.

a. How is the value of b related to the period of the function?

b. What is the value of b for the function that models the motion of the Ferris wheel? Explain.

9. Use the values from your answers in Items 6–8 to write an equation in the form $f(t) = a \sin b(t - h) + k$.

MATH TIP

The graph of the equation $y = a \sin b\,(x - h) + k$ has amplitude $|a|$ and midline $y = k$.

Check Your Understanding

10. How can you check that the equation you wrote in Item 9 is reasonable?

11. How does the graph of the function $f(t)$ compare to the graph of the parent function, $y = \sin x$? Use the language of transformations in your answer.

12. Give a reasonable domain and range of the function $f(t)$.

13. **Make sense of problems.** Tyrell uses the function to make some predictions about the position of the car at various times.

a. What does $f(20)$ represent?

b. What is the value of $f(20)$ to the nearest tenth?

c. How do you know that the value you found for $f(20)$ is reasonable?

My Notes

14. During the first two complete rotations of The Sky Wheel, give the times when the car is moving upward and the times when the car is moving downward.

15. Use appropriate tools strategically. The Sky Wheel will be part of an amusement park that has several pine trees that are 80 feet tall. Tyrell wants to know the times when the car will be above the height of the treetops.

a. Write an equation that Tyrell can solve to find the times when the car will be at the same height as the treetops.

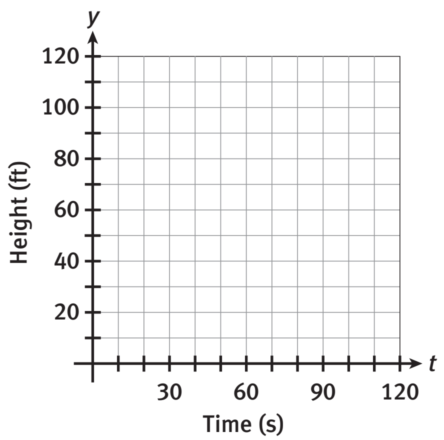
b. Use your calculator to find the times when the car will be at the same height as the treetops during the first two complete rotations of The Sky Wheel. Round to the nearest tenth of a second.

c. During what intervals of time will the car be above the height of the treetops?

16. Tyrell wants to write a new function, $g(t)$, that models the motion of a car that starts at the bottom of The Sky Wheel (point D in the figure at the beginning of the lesson).

a. Modify the function $f(t)$ you wrote in Item 9 to write the new function $g(t)$.

b. Sketch the graph of $g(t)$ below.



My Notes

Check Your Understanding

17. How are the graphs of $f(t)$ and $g(t)$ similar? How are they different?

18. Construct viable arguments. Suppose the client decides that the highest point of The Sky Wheel should be 102 feet rather than 100 feet. The size of the wheel does not change. How would the function $f(t)$ need to be changed?

Tyrell wants to compare The Sky Wheel to some other Ferris wheels he has designed. He checks his files for information on The Round Robin and The Spin Cycle. The information he finds for these Ferris wheels is shown below.

The Round Robin

Time (seconds)	0	12	24	36	48	60
Height (feet)	8	45	82	45	8	45

The Spin Cycle

$$y = 33\cos\left(\frac{\pi}{45}t\right) + 43$$

19. Which of the three Ferris wheels (The Sky Wheel, The Round Robin or The Spin Cycle) rotates the fastest? Justify your answer.

20. Which of the three Ferris wheels is the tallest? Justify your answer.

21. Which of the three Ferris wheels has the greatest diameter? Justify your answer.

CONNECT TO AP

Periodic functions can be modeled by functions other than trigonometric functions. For example, the piecewise-defined function $f(x)$ shown below is periodic but not trigonometric.

$$f(x) = \begin{cases} 1 & x \text{ is an integer} \\ 0 & x \text{ is not an integer} \end{cases}$$

In calculus, you will explore these types of periodic functions when you study limits.

My Notes

Check Your Understanding

22. Explain how you can determine the height of the car on The Round Robin after 4 minutes.

23. Given the trigonometric equation that models the motion of a Ferris wheel, how can you determine the diameter of the Ferris wheel?

LESSON 35-1 PRACTICE

24. Which of the three Ferris wheels in the activity (The Sky Wheel, The Round Robin, or The Spin Cycle) comes closest to the ground? What is this Ferris wheel's distance from the ground?

25. **Persevere in solving problems.** Refer to the table on the previous page that describes the motion of a car on The Round Robin.

a. Write an equation that models the motion of the car.

b. What is the height of the car after 15 seconds?

c. During the first complete rotation of The Round Robin, at what time(s) is the car at a height of 65 feet? Round to the nearest tenth of a second.

26. Tyrell proposes to the client that The Sky Wheel rotate clockwise instead of counterclockwise. How should Tyrell modify the function $f(t)$ to model a clockwise rotation of the Ferris wheel?

27. The equation $y = 4\sin\left(\frac{\pi}{4}(x - 4)\right) + 7$ models the motion of a point on the edge of a circular gear, where x is the number of seconds since the gear started turning and y is the height of the point above the ground, in inches.

a. How long does it take for the gear to make one complete rotation?

b. What is the height of the point after 6 seconds?

28. **Model with mathematics.** The tide at a dock has a minimum height of 0.3 feet and a maximum height of 6.9 feet. It takes a total of 12.2 hours for the tide to come in and go back out. A fisherman wants to model the height h of the tide in feet as a trigonometric function of the time t in hours.

a. What is the period of the function? What is the amplitude?

b. Assume the low tide occurs at time $t = 0$. Write a function of the form $f(t) = a\sin b(t - h) + k$ that models the tide.

c. Explain how you can check that your function is reasonable.

d. What is the value of $f(15)$ to the nearest tenth? What does this represent?

ACTIVITY 35 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 35-1

1. A Ferris wheel has a diameter of 94 feet, and the highest point of the wheel is 102 feet above the ground. The Ferris wheel makes one rotation every 80 seconds.

a. Write a trigonometric function that models the motion of one car on the Ferris wheel.

b. According to your model, what is the height of the car when the ride starts?

c. What is the height of the car after 4 seconds?

2. A bicycle wheel has a diameter of 26 inches. Isabelle rides the bike so that the wheel makes two complete rotations per second. Which function models the height of a spot on the edge of the wheel?

A. $h(t) = 13\sin(2\pi t) + 13$

B. $h(t) = 13\sin(4\pi t)$

C. $h(t) = 13\sin(4\pi t) + 13$

D. $h(t) = 13\sin(2\pi t)$

3. The function $f(x)$ models the height in feet of the tide at a specific location x hours after high tide.

$$f(x) = 3.5\cos\left(\frac{\pi}{6}x\right) + 3.7$$

a. What is the height of the tide at low tide?

b. What is the period of the function? What does this tell you about the tides at this location?

c. How many hours after high tide is the tide at a height of 3 feet for the first time?

4. An office building has a large clock on one face of the building. The minute hand of the clock is 12 feet long, and the center of the clock is 160 feet above the ground. The function $f(t)$ models the height of the tip of the minute hand above the ground in feet, with t representing the time in minutes.

a. What is the period of the function?

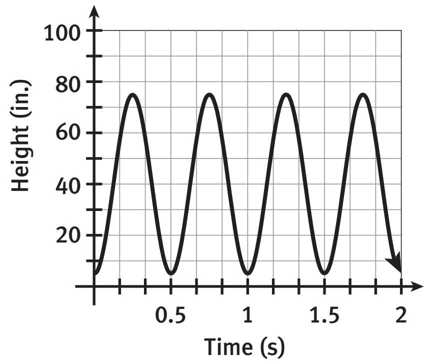
b. Write an equation for $f(t)$ in the form $f(t) = a\sin b(t - h) + k$. Assume the minute hand points to the 12 on the clock at $t = 0$. (*Hint:* Be sure to write the function so that the minute hand is rotating clockwise!)

c. Graph the function.

d. What is the value of $f(15)$? Explain why this makes sense.

e. Explain how you can find $f(180)$ without using a calculator.

5. Sonia and Jeremy turn a jump rope. The graph shows the height of a point at the middle of the rope. How many times do Sonia and Jeremy turn the rope each minute?



For Items 6–10, the height of an object, in centimeters, is modeled by the function $y = 42\sin\left(\frac{\pi}{10}(x-h)\right) + 55$. Determine whether each statement is always, sometimes, or never true.

6. The period of the function is 20.

7. The maximum height of the object is 55 centimeters.

8. The minimum height of the object occurs when $x = 0$.

9. The graph of the function has the midline $y = 55$.

10. The amplitude of the function is 84.

11. The function $f(t) = 40\sin\left(\frac{\pi}{45}t\right) + 48$ models the height in feet of one car of a Ferris wheel called The Colossus, where t is the time in seconds. Each of the functions below models the motion of a different Ferris wheel. Which Ferris wheel has the same diameter as The Colossus?

A. $g(t) = 40\cos\left(\frac{\pi}{45}t\right) + 50$

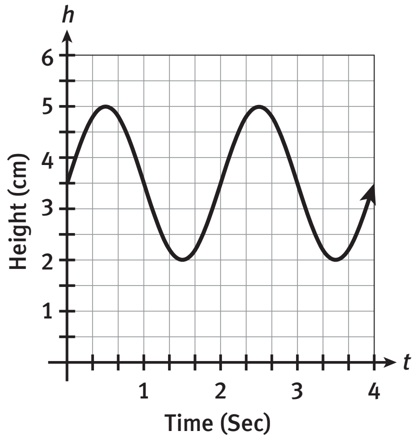
B. $h(t) = 39\cos\left(\frac{\pi}{60}t\right) + 49$

C. $j(t) = 39\sin\left(\frac{\pi}{45}t\right) + 48$

D. $k(t) = 39\sin\left(\frac{\pi}{45}t\right) + 49$

12. The motion of a point on the drum of a clothes dryer is modeled by the function $y = 12\sin\left(\frac{4}{3}\pi t\right) + 20$, where t is the time in seconds. How many times does the dryer rotate per minute?

13. The graph shows the height of a scratch on the edge of a circular gear.



Which function is the best model for the height of the scratch?

A. $h(t) = 3.5\sin(\pi t) + 1.5$

B. $h(t) = 1.5\sin(\pi t) + 3.5$

C. $h(t) = 1.5\sin(2\pi t) + 3.5$

D. $h(t) = 1.5\sin\left(\frac{\pi}{2}t\right) + 3.5$

14. The height in feet of an object above the ground is modeled by the function $y = 3\cos(3\pi t) + 7.8$, where t is the time in minutes. During the first complete cycle, at what times is the object closer than 6 feet to the ground? Use an inequality to express your answer.

MATHEMATICAL PRACTICES
Construct Viable Arguments and Critique the Reasoning of Others

15. A student was asked to model the motion of one car of a Ferris wheel. The student claimed that it is possible to use the tangent function to model the motion, since the tangent function is periodic. Do you agree or disagree with the student's reasoning? Justify your answer.

Totally Tires sells tires and wheels for everything from bicycles to monster trucks. The front outside wall of the shop features a rotating tire to attract customers. The shop's owner has decided that the display needs to be improved. You have been hired to analyze the motion of the existing tire and to add another rotating tire to the display.

1. According to the shop's owner, the motion of the existing tire can be modeled by the function $y = \cos\left(\frac{\pi}{15}x\right) + 5$, where x is the time, in seconds, and y is the height, in feet, of a point on the edge of the tire.
   a. What are the reasonable domain and range of the function?
   b. What is the period of the function? What does this represent?
   c. Graph the function.
   d. How does the graph compare to the graph of the parent function, $y = \cos x$?
2. The shop's owner states that the new rotating tire that will be added to the display should have a diameter of 3 feet and that the top of the tire should be 7 feet above the ground. The owner would like this tire to make one complete rotation every 20 seconds. Write a function of the form $f(t) = a \sin b(t - h) + k$ to model the motion of a point on the edge of the new tire.
3. **Reason abstractly.** Before the new tire is added to the display, the shop's owner wants a written statement so she can get a sense of what the completed display will look like. Write a summary comparing the existing rotating tire to the new one. Be sure to include answers to the following questions, with justifications.
   - Which tire rotates more quickly?
   - Which tire has a greater diameter?
   - Which tire comes closer to the ground?

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
	The solution demonstrates these characteristics:			
Mathematics Knowledge and Thinking (Items 1, 2)	• Clear and accurate understanding of sinusoidal models including domain and range	• Largely correct understanding of sinusoidal models including domain and range	• Partial understanding of sinusoidal models including domain and range	• Incomplete or inaccurate understanding of sinusoidal models including domain and range
Problem Solving (Item 3)	• An appropriate and efficient strategy that results in a correct answer	• A strategy that may include unnecessary steps but results in a correct answer	• A strategy that results in some incorrect answers	• No clear strategy when solving problems
Mathematical Modeling / Representations (Items 1, 2, 3)	• Effective understanding of how features of an equation or graph relate to a real-world scenario • Fluency in applying transformations to graph a function • Fluency in writing a sinusoidal equation to model a real-world scenario	• Mostly accurate recognition of how features of an equation or graph relate to a real-world scenario • Little difficulty in applying transformations to graph a function • Little difficulty in writing a sinusoidal equation to model a real-world scenario	• Partial recognition of how features of an equation or graph relate to a real-world scenario • Some difficulty in applying transformations to graph a function • Some difficulty in writing a sinusoidal equation to model a real-world scenario	• Difficulty in recognizing how features of an equation or graph relate to a real-world scenario • Significant difficulty in applying transformations to graph a function • Significant difficulty in writing a sinusoidal equation to model a real-world scenario
Reasoning and Communication (Items 1d, 3)	• Precise use of appropriate math terms and language when interpreting and comparing models • Clear and accurate description of transformations of a parent function	• Adequate use of math terms and language when interpreting and comparing models • Adequate description of transformations of a parent function	• Misleading or confusing use of math terms and language when interpreting and comparing models • Misleading or confusing description of transformations of a parent function	• Incomplete or mostly inaccurate use of appropriate math terms and language when interpreting and comparing models • Incomplete or inadequate description of transformations of a parent function