Radical and Rational Functions

5

Unit Overview

In this unit, you will extend your study of functions to radical, rational, and inverse functions. You will graph radical and rational functions using transformations and by analyzing key features of the graph, and you will examine the domain and range of the functions. You will solve rational equations and inequalities as well as equations with rational exponents. You will also solve inverse and combined variation problems, average cost per unit problems, and work problems that are modeled using rational functions.

Key Terms

As you study this unit, add these and other terms to your math notebook. Include in your notes your prior knowledge of each word, as well as your experiences in using the word in different mathematical examples. If needed, ask for help in pronouncing new words and add information on pronunciation to your math notebook. It is important that you learn new terms and use them correctly in your class discussions and in your problem solutions.

Math Terms

- square root regression
- one-to-one function
- rational function
- horizontal asymptote
- vertical asymptote
- inverse variation
- constant of variation
- combined variation
- joint variation
- complex fraction
- discontinuity
- removable point of discontinuity

ESSENTIAL QUESTIONS

- Why is it important to consider the domain and range of a function?
- How are rational functions useful in everyday life?

EMBEDDED ASSESSMENTS

This unit has three embedded assessments, following Activities 26, 28, and 30. The first will give you the opportunity to demonstrate what you have learned about radical functions and their inverses. The second assessment focuses on inverse and combined variation. You will also graph rational functions using transformations of the parent function, and you will use rational functions to model average cost per unit. In the third assessment, you will graph rational functions by analyzing key features, such as asymptotes and intercepts, and you will solve rational equations and inequalities.

UNIT 5

Getting Ready

Write your answers on notebook paper.
Show your work.

1. Evaluate each of the expressions.

a. $3\sqrt{49}$

b. $2\sqrt[3]{64}$

c. $\left(\sqrt{x+2}\right)^2$

2. Perform the indicated operation.

a. $\frac{2x}{5} - \frac{3x}{10}$

b. $\frac{2x+1}{x+3} + \frac{4x-3}{x+3}$

c. $\frac{2}{x} + \frac{5}{x+1}$

d. $\frac{2x}{7} \cdot \frac{21}{x^2}$

e. $\frac{x^3}{6} \div \frac{x}{12}$

3. Simplify each expression.

a. $(2x^2y)(3xy^3)$

b. $(4ab^3)^2$

c. $\frac{16x^3}{4x}$

d. $\frac{2x+12}{x+6}$

4. What values are not possible for the variable x in each expression below? Explain your reasoning.

a. $\frac{2}{x}$ **b.** $\frac{2}{x-1}$

5. Factor each expression.

a. $81x^2 - 25$

b. $2x^2 - 5x - 3$

6. Which of the following is the inverse of $h(x) = 3x - 7$?

A. $7 - 3x$ **B.** $3x + 7$

C. $\frac{x+7}{3}$ **D.** $\frac{1}{3x-7}$

7. Write each inequality in interval notation.

a. $x > -5$

b. $x \leq 2$

c. $-3 < x \leq 7$

8. If y varies directly as x and $y = 24$ when $x = 16$, what is y when $x = 50$?

Square Root and Cube Root Functions

Go, Boat, Go!

Lesson 25-1 Square Root Functions

My Notes

Learning Targets:

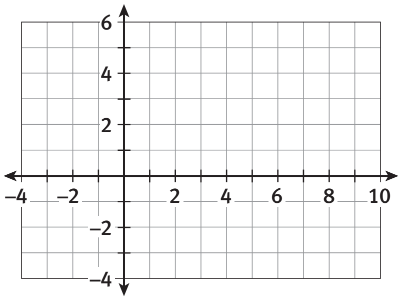
- Graph and describe transformations of the square root function $y = \sqrt{x}$.
- Interpret key features of a graph that models a relationship between two quantities.

SUGGESTED LEARNING STRATEGIES: Create Representations, Note Taking, Think-Pair-Share, Look for a Pattern, Work Backward

The hull speed H, in knots, of a boat is given by the function $H(x) = 1.34\sqrt{x}$, where x is the length of the boat in feet at the waterline.

1. The hull speed function is a transformation of the parent square root function $f(x) = \sqrt{x}$.

a. Graph H and f on the same axes. How do these graphs compare to each other?



b. What are the domain and the range of f? Write your answers as inequalities, in set notation, and in interval notation.

c. **Model with mathematics.** Given that x represents the length of the boat, should the domain of $H(x)$ be more restricted than $f(x)$? Can you determine the domain precisely? Explain your reasoning.

CONNECT TO TRANSPORTATION

The speed of a boat is measured in knots (nautical miles per hour). The distance it travels in water is measured in nautical miles. A nautical mile is equal to 1.15 statute miles.

MATH TIP

To graph the parent square root function, use key points with x-values that are perfect squares, such as 0, 1, 4, and 9.

My Notes

MATH TIP

Recall that the function $y = a \cdot f(x)$ represents a *vertical stretch or shrink* of the original function $y = f(x)$ after the *y*-values have been multiplied by *a*.

MATH TIP

Recall that the function $y = f(x \pm c)$ results in a *horizontal translation* of the original function while $y = f(x) \pm c$ results in a *vertical translation* of the original function.

2. Explain how you could use transformations of the graph of $f(x) = \sqrt{x}$ to graph $g(x) = 2\sqrt{x}$.

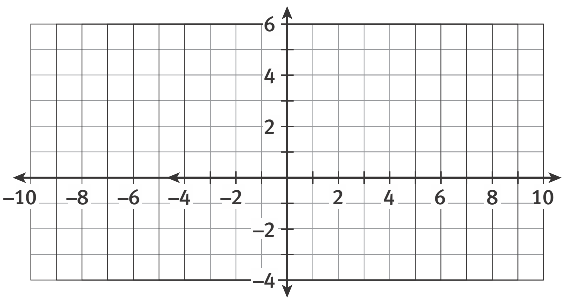
3. Consider the functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2x}$.

a. Write $g(x)$ as a product in the form $a\sqrt{x}$.

b. Explain how you could use transformations of the graph of $f(x) = \sqrt{x}$ to graph $g(x) = \sqrt{2x}$.

4. How does the graph of $g(x) = \sqrt{-x-3}$ compare to the graph of $f(x) = \sqrt{x}$?

5. Sketch *g* and *f* from Item 4 on the same axes below.



6. What are the domain and range of *g*?

Check Your Understanding

7. What does the graph in Item 1 tell us about the relationship between the length of a boat and its hull speed?

8. The *x*-intercept of the parent function $f(x) = \sqrt{x}$ is (0, 0). Without using transformations, how would you find the *x*-intercept, also known as the root, of $f(x) = \sqrt{x-3}$?

My Notes

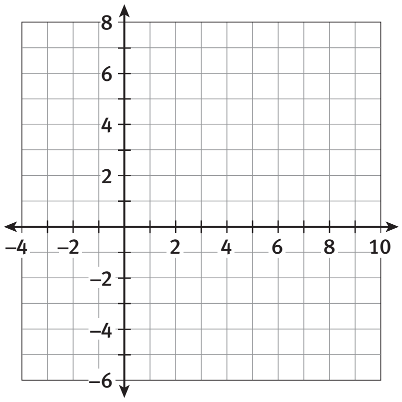
Multiple transformations can be applied to the basic function to create a new function. Transformations might include translations, reflections, stretching, or shrinking.

9. Describe the transformations of $f(x) = \sqrt{x}$ that result in the functions listed below.

a. $g(x) = -\sqrt{x+2}$

b. $h(x) = \sqrt{x-3} + 4$

10. Sketch the graph of each function in Item 9 as well as the parent function. Use a calculator to check your results. Then state the domain and range for each function. Write your answers as inequalities, in set notation, and in interval notation.



11. Without graphing, determine the domain and range of the function $f(x) = \sqrt{x+5} - 1$.

MATH TIP

Recall that the function $y = -f(x)$ represents a *reflection over the x-axis* of the original function $y = f(x)$ after the y-values have been multiplied by -1.

TECHNOLOGY TIP

One way to enter a square root equation into a graphing calculator is to write it using a fractional exponent. Recall that $\sqrt{x}$ can be written as x raised to the $\frac{1}{2}$ power. So, you could enter $\sqrt{x-3} + 4$ as $(x-3)^{\left(\frac{1}{2}\right)} + 4$. Make sure to place parentheses around the fractional exponent.

Check Your Understanding

12. Describe $f(x) = 2\sqrt{x-3}$ as a transformation of $f(x) = \sqrt{x}$. State the domain and range.

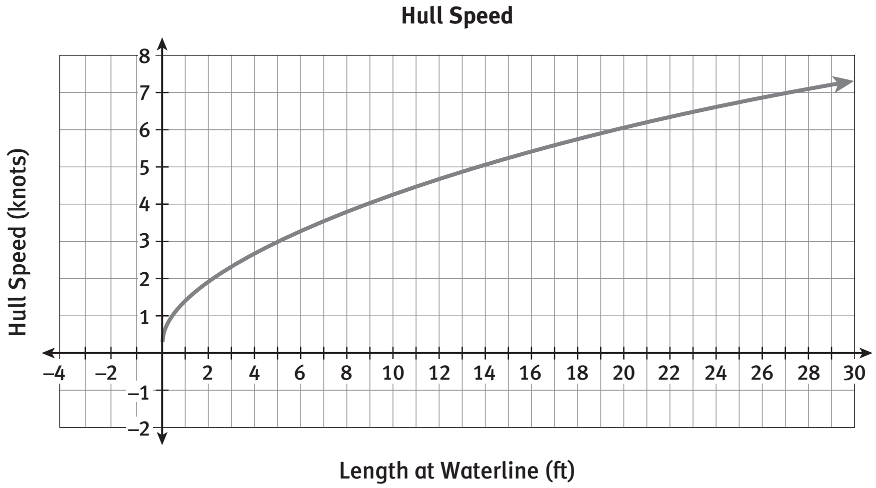
13. Graph $f(x) = \sqrt{x+2} - 1$ using your knowledge of transformations.

14. Give a transformation of the square root function that has a range that approaches negative infinity as x approaches infinity.

15. Use the graph of $h(x)$ in Item 10 to make a conjecture about the solution of the equation $\sqrt{x-3} + 4 = 0$.

My Notes

The graph of the hull speed of a boat H is shown below. You also sketched this graph in Item 1a.



16. Use the graph to estimate the hull speed of a boat that is 24 feet long at the waterline.

17. Use the graph to estimate the length at the waterline of a boat whose hull speed is 6 knots.

18. Write an equation that could be solved to determine the length at the waterline of a boat with a hull speed of 6 knots.

Check Your Understanding

19. Use the graph at the top of the page to estimate the hull speed of a boat that is 9 feet long at the waterline.

20. Explain how you can tell from the graph above that the equation relating the hull speed and the length of the boat is not $H(x) = \sqrt{x}$.

LESSON 25-1 PRACTICE

21. Graph $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{2}\sqrt{x}$ on the same axes.

22. Describe $g(x)$ as a transformation of $f(x)$. What are the domain and range of each function?

23. Graph $p(x) = \sqrt{x}$ and $q(x) = \sqrt{x+4} - 2$ on the same axes.

24. Describe $q(x)$ as a transformation of $p(x)$. What are the domain and range of each function?

25. **Reason abstractly.** Write a square root function that has a domain of $x \geq 7$ and a range of $y \geq 2$. Use a graphing calculator to confirm that your function meets the given requirements.

My Notes

Learning Targets:

- Solve square root equations.
- Identify extraneous solutions.

SUGGESTED LEARNING STRATEGIES: Note Taking, Identify a Subtask, Marking the Text, Predict and Confirm, Create Representations

To solve *square root equations*, follow these steps.

Step 1: Isolate the radical term.

Step 2: Square both sides of the equation.

Step 3: Solve for the unknown(s).

Step 4: Check for extraneous solutions.

MATH TIP

An *extraneous solution* can be introduced when you square both sides of an equation to eliminate the square root. The resulting equation may not be equivalent to the original for all values of the variable.

Example A

Solve the equation $\sqrt{x-3}+4=9$.

$$\sqrt{x-3}+4=9$$

Step 1: Isolate the radical. $\sqrt{x-3}=5$

Step 2: Square both sides. $(\sqrt{x-3})^2=(5)^2$

Step 3: Solve the equation. $x-3=25$, so $x=28$

Step 4: Check the solution.

$$\sqrt{28-3}+4\stackrel{?}{=}9$$
$$5+4=9$$

Example B

Solve the equation $x=(x+1)^{\frac{1}{2}}+5$.

Step 1: Isolate the radical.

$$x=(x+1)^{\frac{1}{2}}+5$$
$$x-5=(x+1)^{\frac{1}{2}}$$

Step 2: Square both sides.

$$(x-5)^2=\left[(x+1)^{\frac{1}{2}}\right]^2$$
$$x^2-10x+25=x+1$$

Step 3: Solve for x.

$$x^2-11x+24=0$$
$$(x-3)(x-8)=0$$

possible solutions $x=3, 8$

Step 4: Check the possible solutions.

$$3\stackrel{?}{=}\sqrt{3+1}+5$$
$$3\neq 2+5$$
$$8\stackrel{?}{=}\sqrt{8+1}+5$$
$$8=3+5$$

Only $x=8$ is a solution; $x=3$ is an extraneous solution.

WRITING MATH

You can write $\sqrt{x}$ as "x to the $\frac{1}{2}$ power," namely, $x^{\frac{1}{2}}$.

My Notes

Try These A–B

Solve each equation.

a. $2-\sqrt{x+1}=-5$

b. $\sqrt{x+4}=x-8$

c. $(x+6)^{\frac{1}{2}}=-x$

d. $(x+4)^{\frac{1}{2}}+1=0$

1. Solve the hull speed equation you wrote in Item 18 of the previous lesson.

2. **Construct viable arguments.** Maggie claims that her sailboat *My Hero* has a hull speed of 7 knots. The length of her boat at the waterline is 24 feet. Is Maggie's claim reasonable? Explain why or why not.

Check Your Understanding

3. Solve each equation.

a. $(x-1)^{\frac{1}{2}}=4$

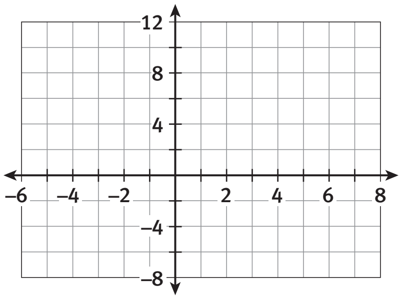
b. $x+\sqrt{2x+3}=0$

4. Solve the equation $0=\sqrt{x+5}$, and then use transformations to sketch the graph of $f(x)=\sqrt{x+5}$. Make a connection between graphing $y=\sqrt{x+5}$ and solving the equation $0=\sqrt{x+5}$.

5. Use your solution to the equation in Try These A–B part d to predict where the graph of $f(x)=(x+4)^{\frac{1}{2}}+1$ will intersect the x-axis. Explain your reasoning.

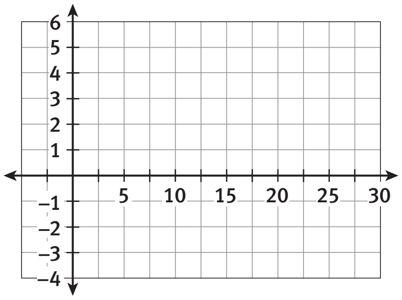
You can also use technology to help you solve equations.

6. Solve the equation $2\sqrt{x+4}=6$ using a graphing calculator. Enter the left side as one function and the right side as another function. Label the point where the graphs intersect.



My Notes

7. Solve the equation $\sqrt{x-1} = 5$ using the same method as in Item 6.



Check Your Understanding

8. Solve the equation $2\sqrt{x+1} = 4$ using a graphing calculator. Include a sketch of the graph to support your answer.

9. In Example B, one of the two possible solutions found to the equation was an extraneous solution. Use a graphing calculator to solve the equation in Example B. Does the graph support the algebraic conclusion? Explain your reasoning.

10. In Item 6, you entered the left side of the equation as one function and the right side as another function and then found the point of intersection. Why is the x-coordinate of the point of intersection the solution to the equation?

LESSON 25-2 PRACTICE

Solve each equation algebraically. Identify any extraneous solutions. Check your solutions using a graphing calculator.

11. $(x-6)^{\frac{1}{2}} = 4$

12. $3\sqrt{x+2} - 7 = 5$

13. $x + \sqrt{x+3} = 3$

14. $\sqrt{x-2} + 7 = 4$

15. Explain how graphing $f(x) = \sqrt{x-2} + 7$ and $g(x) = 4$ supports your algebraic solution to the equation in Item 14.

16. **Make sense of problems.** The approximate intersection of the graphs of $f(x) = \sqrt{x+7}$ and $g(x) = x - 1$ is (4.4, 3.4). Therefore, $x = 4.4$ is the approximate solution to what equation?

My Notes

Learning Targets:

- Graph transformations of the cube root function $y = \sqrt[3]{x}$.
- Identify key features of a graph that models a relationship between two quantities.

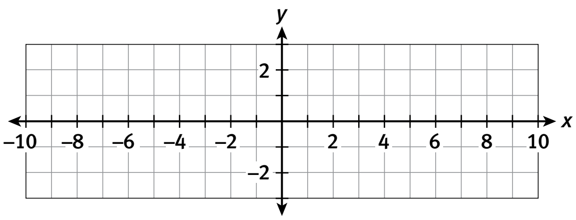
SUGGESTED LEARNING STRATEGIES: Create Representations, Note Taking, Look for a Pattern

The function $r(x) = \sqrt[3]{\frac{3}{4\pi}x}$ represents the length of the radius of a sphere as a function of its volume, represented here by x. An approximation of this function is $r(x) = \sqrt[3]{0.24x}$.

1. The radius function is a transformation of the parent cube root function $f(x) = \sqrt[3]{x}$.

a. Write $r(x)$ as a product in the form $r(x) = a\sqrt[3]{x}$ or $r(x) = ax^{\frac{1}{3}}$ so that it is easier to see the relationship between it and the parent function. Round a to one decimal place.

b. Graph $r(x)$ and $f(x)$ on the same axes. How do these graphs compare to each other? How would the graphs of $h(x) = x^3$ and $j(x) = 0.6x^3$ compare?



c. What are the domain and the range of $f(x)$? Of $r(x)$? Write your answers as inequalities, in set notation, and in interval notation.

d. Describe the transformations of $h(x) = x^3$ that result in the following functions.

i. $j(x) = (-2x)^3$

ii. $k(x) = (x - 5)^3$

iii. $m(x) = x^3 - 4$

iv. $n(x) = 2(0.5x + 4)^3 - 6$

MATH TIP

The function for the radius used here is the inverse of the formula for the volume of a sphere, $V = \frac{4}{3}\pi r^3$, with V represented by x.

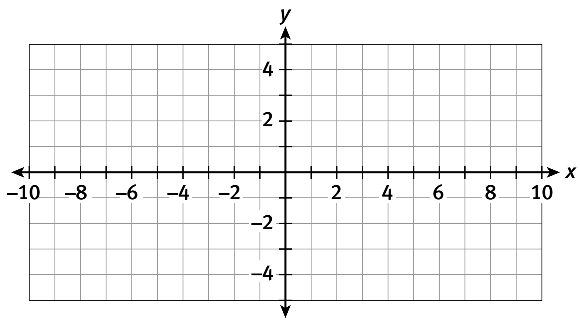
MATH TIP

$\sqrt[3]{x}$ can be written as "x to the $\frac{1}{3}$ power," or $x^{\frac{1}{3}}$.

My Notes

2. How does the graph of $g(x) = \sqrt[3]{-x+1} + 3$ compare to the graph of $f(x) = \sqrt[3]{x}$? How would the graphs of $h(x) = x^3$ and $j(x) = (-x+1)^3 + 3$ compare?

3. Sketch $g(x)$ and $f(x)$ from Item 2 on the same axes below.



4. What are the domain and range of each function?

Check Your Understanding

5. Why is the domain of a cube root function all real numbers, while the domain of a square root function is restricted to only nonnegative numbers?

6. **Attend to precision.** In Item 1, x represents the volume of a sphere and $r(x)$ the volume. Given this context, should the domain and range of $r(x)$ be restricted? Explain your reasoning.

7. Write an equation that could be solved to find the volume of a sphere that has a radius of 2 inches. Write your equation two ways using a rational exponent and a radical.

8. Describe the transformations of $f(x) = \sqrt[3]{x}$ and $h(x) = x^3$ that result in the functions listed below.

a. $g(x) = -\sqrt[3]{x-3}$ and $j(x) = -(x-3)^3$

b. $p(x) = 4\sqrt[3]{-x} - 1$ and $q(x) = 0.25(-4x)^3 - 1$

9. Without graphing, state the domain and range of the functions in Item 8.

My Notes

Check Your Understanding

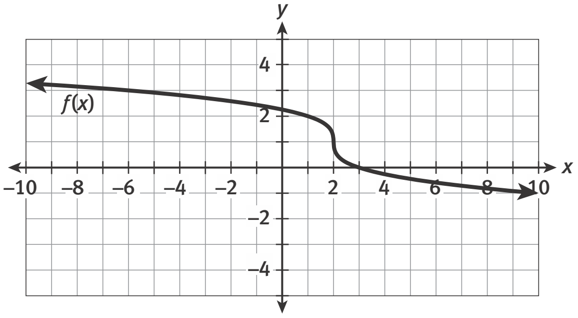
10. Describe the transformations of $f(x) = \sqrt[3]{x}$ that result in the function $f(x) = 2\sqrt[3]{x+4} - 7$. Also, describe the transformations of $h(x) = x^3$ that result in the function $h(x) = 2(x+4)^3 - 7$.

11. **a.** Write an equation for a cube root function that has been reflected across the x-axis and shifted horizontally 5 units to the right.
b. Write an equation for a cubic function that has been reflected across the x-axis and shifted horizontally 5 units to the right.

LESSON 25-3 PRACTICE

12. Use your graph in Item 1b to estimate the radius of a sphere that has a volume of 6 cubic units.

13. Assuming the graph below has not undergone a stretch or a shrink, write a possible equation for the function shown.



14. Sketch the graph of $h(x) = \sqrt[3]{x} - 4$. Describe the transformations. Then state the domain and range.

15. Sketch the graph of $p(x) = \sqrt[3]{x+5} + 2$. Describe the transformations. Then state the domain and range.

16. Consider the statement below.

If a cube root function is reflected across the x-axis and then across the y-axis, the resulting graph is the same as the graph of the original cube root function before the transformations.

Do you agree with the statement? If not, explain why. If you agree, write an algebraic expression that represents the relationship described in the statement.

17. **Make use of structure.** If you solve the equation $\sqrt[3]{x} = 0$, you find that the graph of $f(x) = \sqrt[3]{x}$ intersects the x-axis at $x = 0$. Where does the graph of $f(x) = \sqrt[3]{x+6}$ intersect the x-axis? Explain your reasoning.

18. Describe the transformations of $f(x) = x^3$ that result in the following functions.

a. $f(x) = (-3x)^3$

b. $f(x) = \left(\frac{1}{3}x\right)^3$

c. $f(x) = 4\left(-\frac{1}{2}x\right)^3 - 5$

My Notes

Learning Targets:

- Solve cube root equations.
- Check the reasonableness of solutions.

SUGGESTED LEARNING STRATEGIES: Note Taking, Identify a Subtask, Marking the Text, Create Representations

The steps for solving *cube root equations* are very similar to those for solving square root equations.

Step 1: Isolate the radical term.

Step 2: Cube both sides of the equation.

Step 3: Solve for the unknown(s).

Example A

Solve the equation $4\sqrt[3]{2x-1}-5=3$.

$$4\sqrt[3]{2x-1}-5=3$$

Step 1: Isolate the radical.

$$4\sqrt[3]{2x-1}=8$$

Add 5 to both sides, and then divide both sides by 4.

$$\sqrt[3]{2x-1}=2$$

Step 2: Cube both sides.

$$\left(\sqrt[3]{2x-1}\right)^3=(2)^3$$

Step 3: Solve the equation.

$$2x-1=8$$

$$2x=9$$

$$x=\frac{9}{2}$$

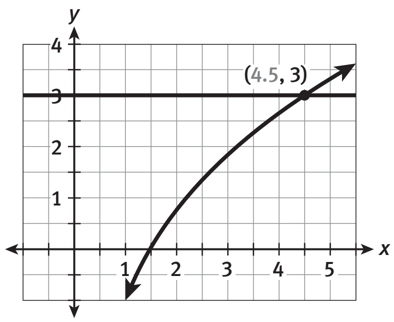
Example B

Use a graphing calculator to solve the equation in Example A.

Step 1: Enter the left side of the equation as one function and the right side as a second function.

Step 2: Find the point of intersection.

Step 3: The x-coordinate of the point of intersection is the solution.



Solution: $x=4.5$

My Notes

Try These A–B

Solve each equation. Use algebraic techniques or a graphing calculator.

a. $4 + (x - 1)^{\frac{1}{3}} = 2$ **b.** $7\sqrt[3]{2x + 5} = 21$

Check Your Understanding

1. Solve each equation.

a. $\sqrt[3]{x - 1} = 5$ **b.** $\sqrt[3]{3x - 1} - 6 = -4$

2. Reason quantitatively. Kari graphed the functions $y_1 = \sqrt[3]{x} - 4$ and $y_2 = \frac{\sqrt[3]{x}}{2}$ on her graphing calculator and found that the intersection of the graphs is (512, 4). What does this tell you about the solution to the equation $\sqrt[3]{x} - 4 = \frac{\sqrt[3]{x}}{2}$?

3. In Item 2, Kari used a graphing calculator to solve the problem. In this particular case, is the use of a graphing calculator more efficient than the use of algebraic techniques? Explain your reasoning.

LESSON 25-4 PRACTICE

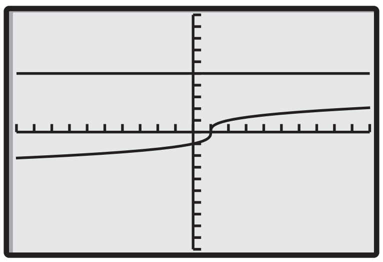
For Items 4–6, solve each equation algebraically. Check your solutions using a graphing calculator.

4. $\sqrt[3]{x} + 3 = 5$

5. $2 + \sqrt[3]{x + 5} = 3$

6. $3\sqrt[3]{2x} = 12$

7. Solving the equation $\sqrt[3]{x - 1} = 5$ using a graphing calculator set to a standard 10-by-10 viewing window yields the following graph.



Does this graph contradict your solution to Item 1a? Explain your reasoning.

8. Critique the reasoning of others. Marcus claims that a softball with a radius of approximately 2 inches has a volume of 40 cubic inches. Is Marcus's claim reasonable? Solve the equation you wrote in Item 7 of the previous lesson to support your answer.

ACTIVITY 25 PRACTICE

Write your answers on notebook paper.
Show your work.

Lesson 25-1

1. The function $L(x)=\sqrt{\frac{x}{6}}$ represents the length of a side of a cube whose surface area is x square units.
   a. Write $L(x)$ as a transformation of the parent square root function $f(x)=\sqrt{x}$.
   b. Graph $L(x)$ and $f(x)$ on the same axes. How do these graphs compare to each other?
   c. The x-intercept of $f(x)$ is $x = 0$. What is the x-intercept of $L(x)$?
   d. What is the domain of $L(x)$? Is the domain reasonable for this scenario?

2. Which of the following square root functions has a domain of $[3, \infty)$?
   A. $f(x)=\sqrt{x}+3$
   B. $f(x)=\sqrt{x}-3$
   C. $f(x)=\sqrt{x+3}$
   D. $f(x)=\sqrt{x-3}$

3. **a.** Explain why the range of $g(x)=-\sqrt{x}$ is $(-\infty, 0]$ rather than $[0, \infty)$. Draw a sketch to support your answer.
   b. Explain why the domain of $h(x)=\sqrt{-x}$ is $(-\infty, 0]$ rather than $[0, \infty)$. Draw a sketch to support your answer.

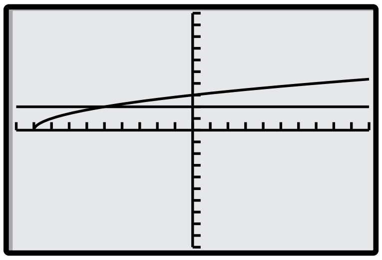
Lesson 25-2

4. **a.** Solve the equation $4=\sqrt{\frac{x}{6}}$ to find the surface area of a cube whose sides are 4 cm long. Show your work.
   b. Use the formula for finding the surface area of a cube, $S = 6s^2$, where s is the length of the side of the cube, to check your results from part a. Is your answer reasonable?

5. Solve each equation algebraically. Identify any extraneous roots.
   a. $\sqrt{x-2}+5=8$
   b. $\sqrt{x+2}+x=0$
   c. $x=\sqrt{3x-12}+4$
   d. $\sqrt{x+5}+2=0$

6. Let $f(x)=\sqrt{x+5}+2$. Sketch the graph of $f(x)$ using what you know about transformations. Identify the x-intercepts, if any, and tell whether your graph supports your answer to Item 5d.

7. The screen shot below shows the solution to a square root equation.



Assuming that the radical term is of the form $\sqrt{x \pm a}$ and that the point of intersection is $(-5, 2)$, write an equation that the graph could be used to solve.

Lesson 25-3

8. Which equation represents the following transformation?

the parent cube root function $f(x) = \sqrt[3]{x}$ *vertically stretched by a factor of 2*

A. $g(x) = \sqrt[3]{8x}$

B. $h(x) = \sqrt[3]{2x}$

C. $p(x) = \sqrt[3]{x-2}$

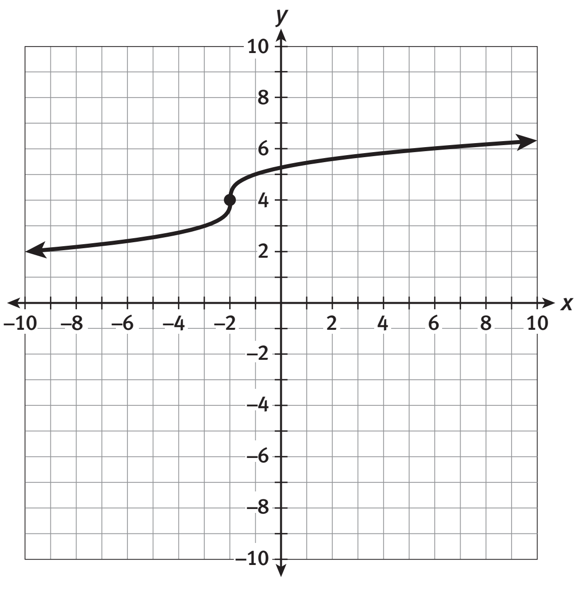
D. $q(x) = \sqrt[3]{x} + 2$

9. Sketch the graph of $f(x) = -\sqrt[3]{x} + 4$ using what you know about transformations.

10. Determine the domain and range of the function $f(x) = a\sqrt[3]{bx - c} + d$. Justify your answer.

11. If possible, give an example of a transformation that changes the domain of a cube root function. If not possible, explain why not.

12. Assuming the graph below represents a cube root function that has not been stretched or shrunk, write a possible equation for the function.



Lesson 25-4

13. Solve each equation using algebraic techniques. Show your work.

a. $\sqrt[3]{x} + 5 = 8$

b. $5\sqrt[3]{x+1} = 10$

14. To enter a cube root function into a graphing calculator, you can write the radical using an exponent of $\frac{1}{3}$. Use this fact to solve the equation $\sqrt[3]{x+2} + 9 = 12$.

MATHEMATICAL PRACTICES
Use Appropriate Tools Strategically

15. You are given the option on a math quiz to solve only one problem using a graphing calculator. You must solve the other problems using algebraic techniques. Which of the following would you choose to solve with the graphing calculator, and why? Be specific. Then solve the equation using a calculator.

Problem 1: $\sqrt{x} + 5 = 17$

Problem 2: $\sqrt{x} + 5 = x$

Problem 3: $\sqrt{x} + x = 5 + x$

Inverses: Roots, Squares, and Cubes

Swing, Swing, Swing

Lesson 26-1 Square Root Functions and Regressions

My Notes

Learning Targets:

- Graph and write the inverse of square root functions.
- Find a square root model for a given table of data.

SUGGESTED LEARNING STRATEGIES: Create Representations, Note Taking, Think-Pair-Share

You have studied linear functions and their inverses as well as logarithmic and exponential functions that are inverse functions. Let's review what we know about inverse functions f and g:

$$f(g(x)) = x \text{ for all } x \text{ in the domain of } g \text{, and}$$
$$g(f(x)) = x \text{ for all } x \text{ in the domain of } f.$$

Now consider one of the radical functions you've just studied: square root functions.

Example A

Graph the inverse of $f(x) = x^{\frac{1}{2}}$. Then give the domain and range of both the function and its inverse.

Step 1: List four points on the graph of f.

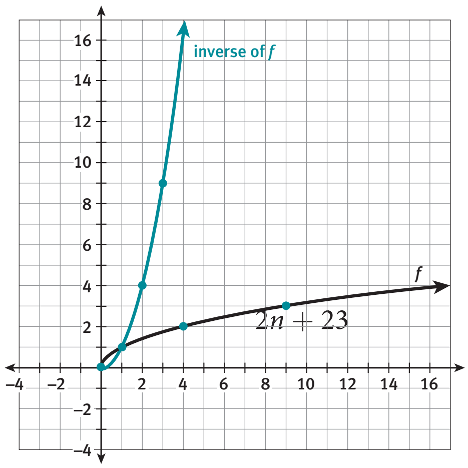
(0, 0), (1, 1), (4, 2), (9, 3)

Step 2: Interchange x and y for each point. These points will be on the graph of the inverse.

(0, 0), (1, 1), (2, 4), (3, 9)

Step 3: Connect the points.

Step 4: Consider the x- and y-values of the graphs to determine the domain and range of the functions.



Solution: The blue graph is the inverse of f. The domain of both the function and its inverse is $x \geq 0$. The range of both the function and its inverse is $y \geq 0$.

Example B

For $f(x) = x^{\frac{1}{2}}$, find f^{-1} algebraically. Then give the domain and range of f^{-1}.

Step 1: Let y represent $f(x)$. $\quad y = x^{\frac{1}{2}}$

Step 2: Interchange x and y to form the inverse relationship. $\quad x = y^{\frac{1}{2}}$

Step 3: Solve for y to find the inverse. $\quad (x)^2 = \left(y^{\frac{1}{2}}\right)^2$

Assume the inverse is a function. $\quad x^2 = y$

Solution: $f^{-1} = x^2$. The domain of f^{-1} is all real numbers and the range is $y \geq 0$.

MATH TIP

Recall that rational exponents are another way of writing radical expressions with

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$
$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

MATH TIP

Recall that if (x, y) is a point on the graph of f, then (y, x) is a point on the graph of its inverse.

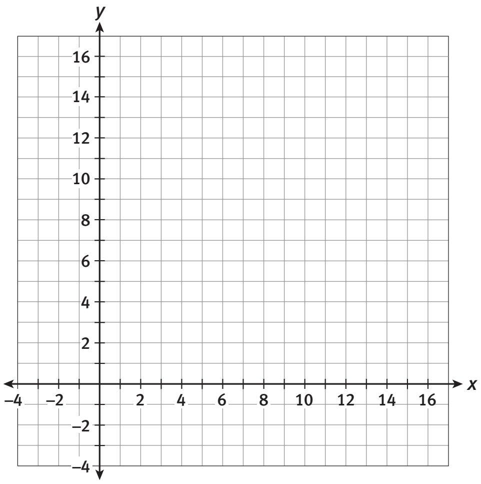
MATH TIP

The typical notation for an inverse function is f^{-1}. It is extremely important to note that $f^{-1}(x) \neq \frac{1}{f(x)}$.

My Notes

Try These A–B

a. Graph $f(x) = (x - 3)^{\frac{1}{2}}$ using the values $x = 3$, 4, and 7. Then graph its inverse on the same axes.



b. For the function in part a, find f^{-1} algebraically.

In Examples A and B, we found the inverse of $f(x) = x^{\frac{1}{2}}$ using two different techniques. In Example A, graphing the inverse resulted in a domain of $x \geq 0$. However, finding the inverse algebraically resulted in the function $f^{-1} = x^2$, which we already know has a domain of all real numbers. The explanation for this is subtle but still very important when defining the inverse of a function: The domain of the inverse function f^{-1} should be restricted to match the range of the original function.

Check Your Understanding

1. Complete this statement about the inverse of $f(x) = x^{\frac{1}{2}}$:
$f^{-1}(x) =$ ____________ for $x \geq$ ____________.

2. Find the inverse of the function $h(x) = 1.34x^{\frac{1}{2}}$.

3. Give the domain and range for both h and h^{-1} in Item 2. Write your answers using inequalities, set notation, and interval notation.

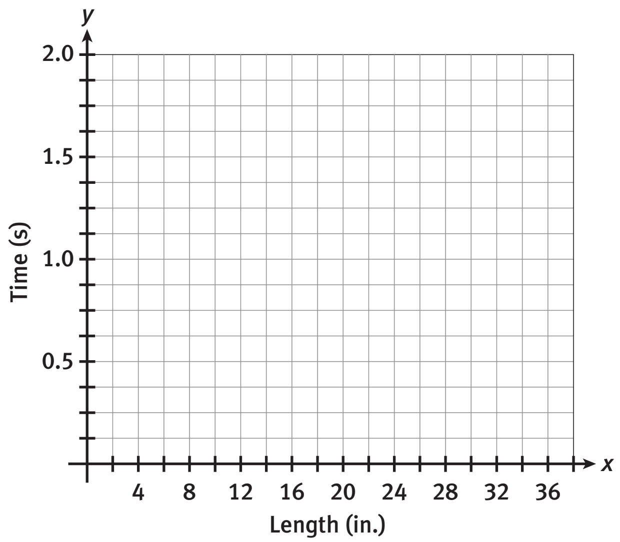
My Notes

In a previous lesson, you learned how to use a graphing calculator to find a quadratic regression based on real world data. Let's do the same for a square root function.

A physics class conducted an experiment comparing the period of a pendulum to the length of the pendulum. The results of the experiment are given in the table below.

Length (in.)	6	10	14	18	22	26	30	34
Time (s)	0.8	1.0	1.2	1.35	1.5	1.6	1.75	1.85

4. Make a scatter plot of the data on the coordinate grid below.



5. What parent function does the path of the data points resemble?

Recall that a quadratic regression is the process of finding a quadratic function that best fits a set of data. We used a graphing calculator to perform quadratic regressions. A ***square root regression*** is a similar process and can also be performed using a graphing calculator.

6. Make use of structure. If a square root function of the form $f(x) = a\sqrt{x}$ is the best fit for the data graphed in Item 4, make a conjecture about the value of a. Explain your reasoning.

CONNECT TO PHYSICS

The period of a pendulum is the length of time it takes to make one cycle, a complete swing back and forth. The period varies with the length of the pendulum, although other factors, which are not taken into account here, can affect the period.

MATH TERMS

A **square root regression** is the process of finding a square root function that best fits a set of data.

My Notes

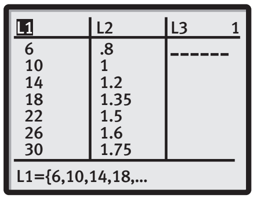
TECHNOLOGY TIP

PwrReg stands for *power regression*. It can be used for any regression of the form $y = ax^b$, when there are no translations.

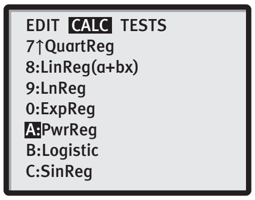
Example C

Use a graphing calculator to perform a regression for the pendulum data and determine the type of function that is the best fit.

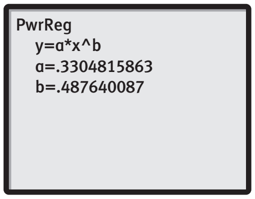
Step 1: Press STAT to open the statistics menu. Choose **Edit** by pressing ENTER. Enter the data: the length data as L1 and the time data as L2.



Step 2: Press STAT again. Move the cursor to highlight **Calc** and then scroll down the submenu to select **A:PwrReg**. Press ENTER.



Step 3: The calculator displays the values of a and b for the standard form of a power function that best fits the data. Use the values of a and b to write the equation. Round all values to two decimal places.



Solution: The equation of the regression is $y = 0.33x^{0.49} \approx 0.33\sqrt{x}$. A square root function is a good fit for the data.

7. Graph the square root model $y = 0.33\sqrt{x}$ on the coordinate grid in Item 4. Does your graph support a "good fit"? Explain your reasoning.

8. Give the domain and range for your square root regression. Write your answers using inequalities, set notation, and interval notation.

9. Use the square root model to predict the period of a pendulum that is 40 inches long. Round your answer to two decimal places.

10. Use the regression equation $y = 0.33\sqrt{x}$ to find the length of a pendulum that has a period of 3 seconds. Round your answer to the nearest half inch.

My Notes

Check Your Understanding

11. The regression equation obtained from the calculator in Example C is $y = 0.33x^{0.49}$. Explain why it is acceptable to rewrite this equation as a square root, $y \approx 0.33\sqrt{x}$.

12. A power regression is only appropriate when there is no horizontal or vertical translation. How can you determine from the context of the pendulum problem that the parent function $f(x) = \sqrt{x}$ has not been translated?

LESSON 26-1 PRACTICE

13. Graph the function $f(x) = 2x^{\frac{1}{2}}$. Then use at least four points from your graph to sketch f^{-1} on the same coordinate grid.

14. Give the domain and range of the function and its inverse in Item 13. Write your answers using inequalities, set notation, and interval notation.

15. Find f^{-1} for the function in Item 13 algebraically. Give restrictions for the inverse if there are any and explain.

16. Consider the data in the table below.

x	2	5	7	10	12	15	18	20
y	4.2	6.7	7.9	9.5	10.4	11.6	12.7	13.4

a. Assuming a square root function in the form $f(x) = a\sqrt{x}$ is a good fit for the data, make a conjecture about the value of *a*. Explain your reasoning.

b. Use a graphing calculator to perform a square root regression. Write your answer using a rational exponent and then using a radical. Round all values to hundredths.

17. **Model with mathematics.** Determine whether a square root function is a good model for the data shown in the table below. Explain your reasoning. If a square root model is not appropriate, offer an alternative model and explain how you arrived at this model.

x	1	2	5	10	12	20	24	31	36
y	2	2.5	3.4	4.3	4.6	5.4	5.8	6.3	6.6

My Notes

Learning Targets:

- Graph and write the inverse of square root functions.
- Find the inverse relations of quadratic functions.

SUGGESTED LEARNING STRATEGIES: Vocabulary Organizer, Create Representations, Quickwrite, Look for a Pattern, Work Backward

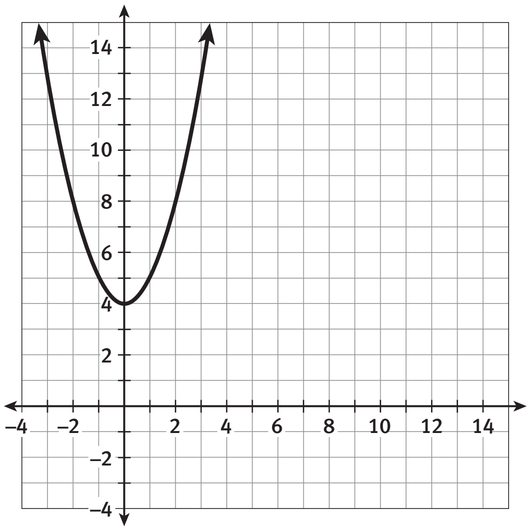
MATH TIP

Recall that a *relation* is a set of ordered pairs that may or may not be defined by a rule. Not all relations are functions, but all functions are relations.

All functions have an inverse *relation*, but the relation may or may not be a function.

1. Use the quadratic function g graphed below.

a. Graph the inverse of g.



b. Is the inverse of g a function? Explain your reasoning.

c. What characteristic of the graph of a function can you use to determine whether its inverse relation is a function?

d. Give the domain and range of the inverse relation in part a. Write your answers using inequalities, set notation, and interval notation.

Example A

The quadratic function shown in the graph in Item 1 is $g(x) = x^2 + 4$. Find an equation for the inverse relation of this function.

Step 1: Let y represent $g(x)$. $\quad y = x^2 + 4$

Step 2: Interchange x and y to form the inverse relationship. $\quad x = y^2 + 4$

Step 3: Solve for y to find the inverse. $\quad x - 4 = y^2$

$$\pm(x-4)^{\frac{1}{2}} = y$$

Solution: $g^{-1}(x) = \pm(x-4)^{\frac{1}{2}} = \pm\sqrt{x-4}$

Try These A

a. Find the inverse of the function $f(x) = -x^{\frac{1}{2}}$. State whether or not the inverse is a function.

b. Find the inverse of the function $g(x) = (x-5)^2$. State whether or not the inverse is a function.

MATH TIP

When taking the square root of a variable, the result yields a positive and a negative value.

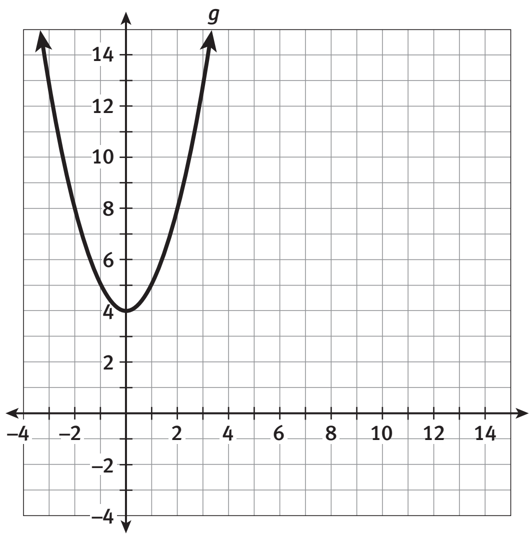
Check Your Understanding

2. In Try These A part a, why is it necessary to include a restriction on the domain in the definition of g^{-1}?

3. What specific part of the equation for g^{-1} in Example A tells you that it is not a function? Explain your reasoning.

If desired, it is possible to restrict the domain of a function so that its inverse will also be a function. Consider the graph of $g(x) = x^2 + 4$ and its inverse from Item 1.

Imagine covering the right side of the graph of g and then drawing the inverse. The result would be only the bottom half of the relation g^{-1}, which would be a function. So if we restrict the domain of g to $x \leq 0$, the inverse of g will be a function with a domain of $x \geq 4$ and a range of $y \leq 0$.



MATH TIP

The range of the inverse of a function is the same as the domain of the original function. You can use this fact to determine whether the inverse of a restricted quadratic function will be the top half or the bottom half of the inverse relation.

My Notes

4. Give another possible restriction on the domain of the function $g(x) = x^2 + 4$ that ensures an inverse that is a function.

Check Your Understanding

State whether the domain of each of the following functions must be restricted to ensure that its inverse is a function. Give an appropriate restriction where needed.

5. $f(x) = \sqrt{x+3}$

6. $g(x) = x^{\frac{1}{2}} + 1$

7. $p(x) = (x - 5)^2$

8. $q(x) = x^2 + 6x + 9$

MATH TIP

The *horizontal line test* is similar to the vertical line test. It is a visual way to determine whether numbers in the range of a function have more than one corresponding number in the domain of the function.

A function is defined as ***one-to-one*** if, for each number in the range of the function, there is exactly one corresponding number in the domain of the function. If $f(x)$ is a one-to-one function, then it will pass both the vertical and *horizontal line test*.

9. Is g from Item 1 a one-to-one function? Explain.

10. **Construct viable arguments.** Make a conjecture about a function whose inverse relation is a function.

Check Your Understanding

Determine whether each type of function will *always*, *sometimes*, or *never* have an inverse that is a function, assuming the domain of the function has not been restricted. Explain your reasoning.

11. Linear function

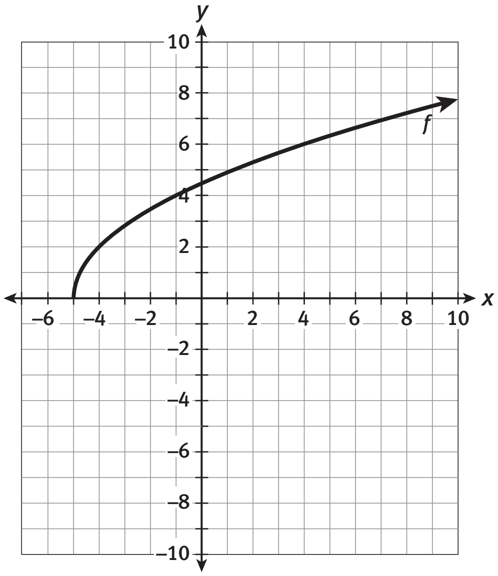
12. Quadratic function

13. Square root function

My Notes

LESSON 26-2 PRACTICE

14. Graph the inverse of the function f on the coordinate grid.



15. In Item 14, $f(x) = 2(x + 5)^{\frac{1}{2}}$. Find f^{-1} algebraically.

16. Find an equation for the inverse relation of $g(x) = 9x^2 - 4$. Then state the domain and range of g and g^{-1}. Write your answers using inequalities, set notation, and interval notation. State if g and g^{-1} are one-to-one.

17. Restrict the domain of the function $h(x) = (x + 7)^2$ so that its inverse will be a function. Then use the restriction to determine whether the graph of the inverse will be the top half or the bottom half of the inverse relation.

18. **Attend to precision.** Explain why an even function can never be a one-to-one function.

MATH TIP

Recall that an *even function* is one in which $f(-x) = f(x)$ and is symmetrical across the y-axis. An *odd function* is one in which $f(-x) = -f(x)$ and is symmetrical around the origin.

My Notes

Learning Targets:

- Graph and write the inverse of cube root functions.
- Find the inverse relations of cubic functions.

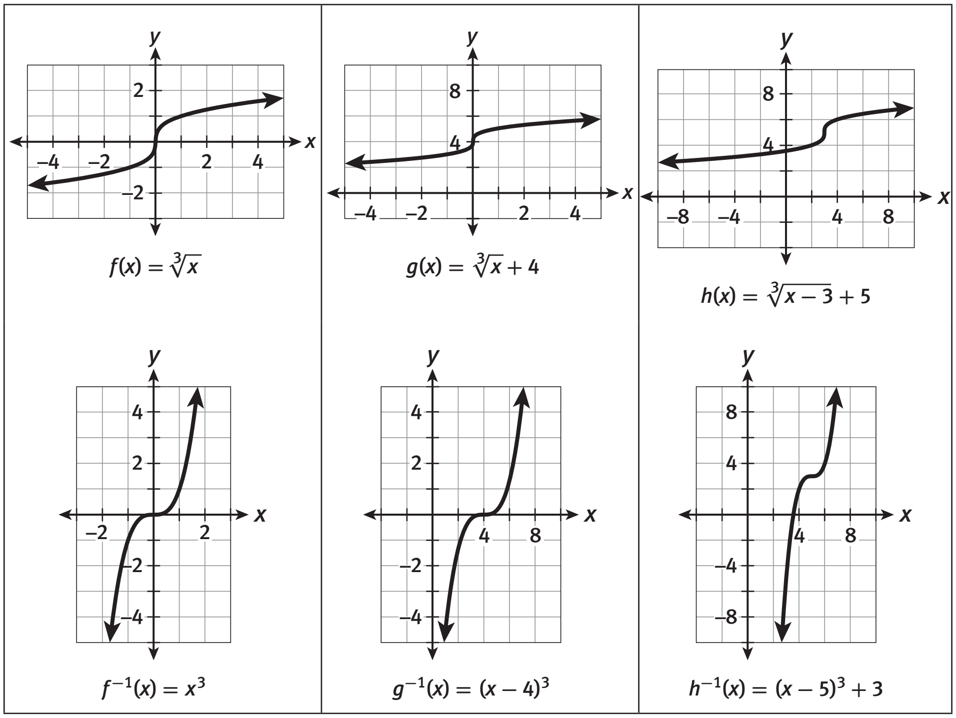
SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Marking the Text

Thus far, we have graphed and found the inverses of linear, exponential, logarithmic, quadratic, and square root functions. We discovered a number of patterns related to these types of functions:

- Linear, exponential, logarithmic, and square root functions always have inverses that are functions.
- Quadratic functions have inverses that are functions only if the domain is restricted. Otherwise, the inverse is a relation only.
- Any function that is one-to-one will have an inverse that is a function. If a function is not one-to-one, the inverse will be a relation only.

Let's explore cube root and cubic functions to see if similar patterns exist. If we can determine whether a given cube root or cubic function is one-to-one, then we will know if the inverse is a function.

Study the graphs below. Each cubic graph in the bottom row is the inverse of the cube root graph above it in the top row. The graphs in the center and right-hand columns are translations of the graphs of the parent functions in the left-hand column.



My Notes

1. Are the cube root functions one-to-one? Are the cubic functions one-to-one? Is each of the inverses a function?

2. Give the domain and range for each of the functions and their inverses. You may write a general statement to describe the domain and range.

3. **Express regularity in repeated reasoning.** Make a conjecture about the inverses of cube root and cubic functions that are translations of the parent functions $f(x) = \sqrt[3]{x}$ and $f(x) = x^3$.

You may have noticed another pattern. The inverse of a translated cube root function is a translated cubic function. Since finding inverses involves interchanging the x- and y-values, the vertical and horizontal translations are also interchanged. Let's verify this with an example.

Example A

Find the inverse of the function $f(x) = \sqrt[3]{x-6} + 1$.

Step 1: Let y represent $f(x)$. $\quad y = \sqrt[3]{x-6} + 1$

Step 2: Interchange x and y. $\quad x = \sqrt[3]{y-6} + 1$

Step 3: Solve for y to find the inverse.

$$x - 1 = \sqrt[3]{y-6}$$
$$(x-1)^3 = y - 6$$
$$(x-1)^3 + 6 = y$$

Solution: $f^{-1}(x) = (x-1)^3 + 6$

Not all cube root or cubic functions are translations of their parent functions, so we need to examine those as well.

Try These A

Find the inverse of the function.

a. $f(x) = \sqrt[3]{x+3}$ **b.** $f(x) = \sqrt[3]{x} + 5$

4. **Use appropriate tools strategically.** Use a graphing calculator to investigate the graph of $f(x) = \sqrt[3]{x^2 + 4x}$ in a standard 10-by-10 viewing window. Sketch the graph. Is f a one-to-one function? Why or why not? Will the inverse of f be a function?

5. Investigate the cubic function $g(x) = x^3 - 6x^2 + 8x + 5$. Use your calculator to graph g in a standard 10-by-10 viewing window. Sketch the graph. Is g a one-to-one function? Will the inverse of g be a function?

My Notes

Check Your Understanding

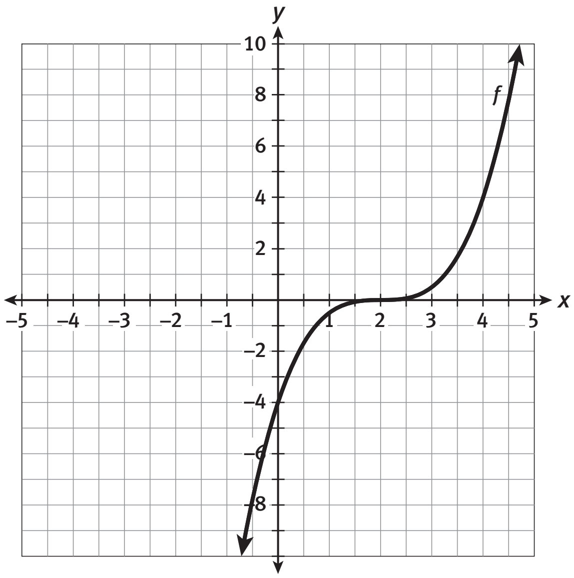
6. Determine without graphing whether the function $f(x) = \sqrt[3]{x} + 2$ is a one-to-one function. Explain your reasoning.

7. Determine without graphing whether the inverse of the function $g(x) = (x + 5)^3$ will be a function. Explain your reasoning.

8. Make a conjecture as to whether cube root functions of the form $f(x) = a\sqrt[3]{x}$ or $f(x) = \sqrt[3]{ax}$ are one-to-one functions and will therefore have inverses that are functions. Explain your reasoning.

LESSON 26-3 PRACTICE

9. The graph below shows the function $f(x) = \frac{1}{2}(x - 2)^3$. Sketch the graph of f^{-1} on the same coordinate grid.



10. Give the domain and range of f and f^{-1}. Write your answers using intervals, set notation, and interval notation.

11. Find the equation of f^{-1} algebraically. Does the domain of f need to be restricted to ensure that f^{-1} is a function? Explain why or why not.

12. Use a graphing calculator to help you determine whether the function $p(x) = x^3 + 2x^2 - 4x + 2$ has an inverse that is a function. Explain how the calculator helps to determine the answer.

13. **Critique the reasoning of others.** Jiao claimed that the cubic function $f(x) = x^3 - 6x^2 + 12x - 8$ is a one-to-one function. She did not use a graphing calculator to make her determination. Steven looked at the function and immediately said Jiao was wrong. Who is correct? Justify your answer without using a graphing calculator.

ACTIVITY 26 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 26-1

1. Magdalena collected the following data for the stopping distances of cars at various speeds.

Stopping Distance (ft)	Speed (mph)
30	25
43	30
59	35
97	45
145	55
172	60
202	65
234	70

She is aware that the data may be modeled by a square root function but she does not know the coefficient of the function.

a. Use the power regression feature of your calculator to find a square root function that fits this data. Round all values to two decimal places.

b. Explain how you could check your regression equation using one of the data points in the table.

c. Write and solve an equation to find the stopping distance of a car that is going 50 mph, assuming the same general road conditions as in Magdalena's study. Show your work.

d. Use the data points in the table to determine if your answer to part c is reasonable. Explain.

2. The domain of a function is $x \geq 2$ and its range is $y \geq 0$. Give the domain and range, using inequalities, of the inverse of the function.

3. The table below shows several points on the graph of $f(x) = 3x^{\frac{1}{2}}$.

x	y
0	0
1	3
4	6
9	9

Use this information to graph f and its inverse on the same set of axes. Then give the domain and range of each using inequalities, set notation, and interval notation.

4. Find the inverse of $g(x) = (3x)^{\frac{1}{2}}$ algebraically.

5. For some function $g(x)$, $g(2) = 5$. Assuming that g^{-1} is a function, which of the following is true?

A. $g^{-1}(2) = \frac{1}{5}$

B. $g^{-1}(2) = -5$

C. $g^{-1}(5) = 2$

D. $g^{-1}(5) = \frac{1}{2}$

Lesson 26-2

6. Give an example of a function whose inverse is not a function.

7. Find the equation of the inverse of the function $h(x) = 5\sqrt{x-1}$. Then state whether the inverse is a function or a relation.

8. Graph the quadratic function $f(x) = x^2 - 4$ and its inverse relation on the same coordinate grid. Then give the domain and range of each using inequalities, set notation, and interval notation.

For Items 9–11, state whether the domain of the function must be restricted to ensure that its inverse is a function. Give an appropriate restriction where needed.

9. $f(x) = (x - 5)^2 + 3$

10. $g(x) = \sqrt{2x + 4}$

11. $h(x) = 4x^2 + 20x + 25$

Lesson 26-3

12. Use a graphing calculator to determine whether $g(x) = x^3 - 2x^2 - 7x - 4$ is a one-to-one function. Then explain what this tells you about the inverse of g.

13. Find the equation of the inverse of the function $f(x) = (2x + 3)^{\frac{1}{3}}$.

14. Draw a sketch to dispute the following statement:

Since an odd function is symmetric about the origin, it will be a one-to-one function.

15. Which of the following cube root functions is NOT one-to-one?

A. $f(x) = \sqrt[3]{3x + 5}$

B. $f(x) = \dfrac{\sqrt[3]{x + 5}}{3}$

C. $f(x) = \sqrt[3]{x + 5} + 3$

D. $f(x) = \sqrt[3]{3x^2 + 5}$

MATHEMATICAL PRACTICES
Reason Abstractly and Quantitatively

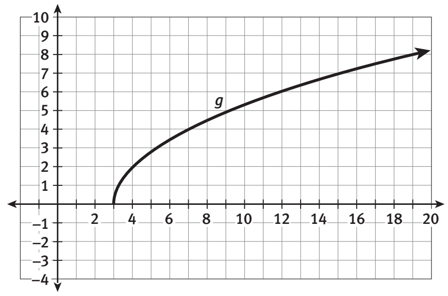
16. When we restrict the domain of a quadratic function to ensure that its inverse will also be a function, we typically use the x-value of the vertex, mainly because it is convenient. However, we can restrict the domain in other ways that will also ensure an inverse that is a function.

For the function $f(x) = x^2 - 8x + 16$, each of the following restrictions, except one, will ensure that the inverse of f will be a function. Circle the restriction that does not ensure that the inverse will be a function. Explain your reasoning.

A. $x \le 0$
B. $x \le 2$
C. $x \le 4$
D. $x \ge 2$
E. $x \ge 4$

Radical Functions: Square Roots, Cube Roots, and Their Inverses

HOW BIG IS THAT BALL?



1. The graph of a function g is shown.

a. Describe the graph as a transformation of $f(x) = \sqrt{x}$.
b. Write the equation for g.
c. State the domain and range of g.
d. Find the inverse of g. Be sure to include any restrictions on the domain of the inverse.
e. Use the graph or a table to solve the equation $g(x) = 7$.

2. Solve the equation $x + \sqrt{x} = 6$.

3. Restrict the domain of the quadratic function $f(x) = x^2 - 2x - 15$ so that f^{-1} is also a function.

4. The data below show the volume, V, of different-sized balls in relation to the length of the radius, r, of the ball.

Ball	Volume (in.^{3})	Radius (in.)
Ping Pong ball	2.07	0.79
Golf ball	2.48	0.84
Racketball	6.04	1.13
Tennis ball	9.20	1.30
Baseball	12.77	1.45
Softball	69.46	2.55
Volleyball	321.56	4.25
Basketball	434.89	4.70

a. Use the power regression feature of your calculator to determine whether a square root or cube root function best fits the data. Enter volume in one list and radius in another list.
b. Write the equation of the regression two ways using a rational exponent and a radical function. Round all values to two decimal places.
c. A cricket ball has a volume of 12.25 cubic inches. Use your regression equation to find the radius of a cricket ball. Round your answer to two decimal places.
d. Most balls are in the general shape of a sphere. The formula for finding the volume of a sphere is $V = \frac{4}{3}\pi r^3$. Find the inverse of V and compare it to your regression equation.

5. The table below gives several points on the graph of a cube root function, $h(x)$.

x	-6	1	2	3	10
y	2	3	4	5	6

a. Use the points in the table to graph the inverse of h.
b. Use what you know about transformations and inverses to write the equation for h^{-1}.
c. Use your equation for h^{-1} to find the equation of the original function, $h(x)$, algebraically.

Radical Functions: Square Roots, Cube Roots, and Their Inverses

HOW BIG IS THAT BALL?

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
	The solution demonstrates these characteristics:			
Mathematics Knowledge and Thinking (Items 1c, 1d, 1e, 2, 3, 4b, 4c, 5)	• Clear and accurate identification of key features of radical functions, including domain and range, and transformations of a parent function • Fluency in solving equations (numerically or graphically, and algebraically) and rewriting expressions containing rational exponents or radicals • Effective understanding of inverse functions, including producing an invertible function by restricting the domain	• Mostly accurate identification of key features of radical functions, including domain and range, and transformations of a parent function • Little difficulty in solving equations (numerically or graphically, and algebraically) and rewriting expressions containing rational exponents or radicals • Largely correct understanding of inverse functions, including producing an invertible function by restricting the domain	• Partially accurate identification of key features of radical functions, including domain and range, and transformations of a parent function • Some difficulty in solving equations (numerically or graphically, and algebraically) and rewriting expressions containing rational exponents or radicals • Partial understanding of inverse functions, including producing an invertible function by restricting the domain	• Incomplete or inaccurate identification of key features of radical functions, including domain and range, and transformations of a parent function • Significant difficulty in solving equations (numerically or graphically, and algebraically) and rewriting expressions containing rational exponents or radicals • Little or no understanding of inverse functions, including producing an invertible function by restricting the domain
Problem Solving (Item 4c)	• An appropriate and efficient strategy that results in a correct answer	• A strategy that may include unnecessary steps but results in a correct answer	• A strategy that results in some incorrect answers	• No clear strategy when solving problems
Mathematical Modeling / Representations (Items 1b, 4a, 4d, 5)	• Effective understanding of how to apply transformations to create an equation from a graph • Fluency in writing expressions for the inverse of a function • Effective understanding of modeling a real-world scenario with a power regression • Clear and accurate understanding of how to rearrange a formula to highlight a quantity of interest	• Largely correct understanding of how to apply transformations to create an equation from a graph • Little difficulty in writing expressions for the inverse of a function • Largely correct understanding of modeling a real-world scenario with a power regression • Mostly accurate understanding of how to rearrange a formula to highlight a quantity of interest	• Partial understanding of how to apply transformations to create an equation from a graph • Some difficulty in writing expressions for the inverse of a function • Partial understanding of modeling a real-world scenario with a power regression • Partially accurate understanding of how to rearrange a formula to highlight a quantity of interest	• Little or no understanding of how to apply transformations to create an equation from a graph • Significant difficulty in writing expressions for the inverse of a function • Little or no understanding of modeling a real-world scenario with a power regression • Incomplete or mostly inaccurate understanding of how to rearrange a formula to highlight a quantity of interest
Reasoning and Communication (Items 1a, 4a, 4d)	• Precise use of appropriate math terms and language to describe a function as a transformation of another function • Clear and accurate description and comparison of equations that model real-world scenarios	• Adequate and largely correct description of a function as a transformation of another function • Adequate description and comparison of equations that model real-world scenarios	• Misleading or confusing description of a function as a transformation of another function • Misleading or confusing description and comparison of equations that model real-world scenarios	• Incomplete or mostly inaccurate description of a function as a transformation of another function • Incomplete or inadequate description and comparison of equations that model real-world scenarios

Introduction to Rational Functions

ACTIVITY 27

Planning a Summer Camp

Lesson 27-1 Formulating and Graphing a Rational Function

My Notes

Learning Targets:

- Formulate rational equations that model real-world situations.
- Graph equations on coordinate axes.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Summarizing, Paraphrasing, Create Representations, Look for a Pattern

The finance committee of a nonprofit summer camp for children is setting the cost for a 5-day camp. The fixed cost for the entire camp is $2400 per day, and includes things such as rent, salaries, insurance, and equipment. An outside food services company will provide meals at a cost of $3 per camper per meal. Campers will eat three meals a day.

As a nonprofit camp, the camp must cover its costs, but not make any profit. The committee must come up with a proposal for setting the fee for each camper, based on the number of campers who are expected to attend each week.

1. Initially, the committee decides to calculate camper fees based on the fixed cost of the camp alone, without meals for the campers.

a. What is the total fixed cost for the five days?

b. Complete the table below to determine the fee per camper that will guarantee the camp does not lose money.

Number of Campers	Fee per Camper

MATH TIP

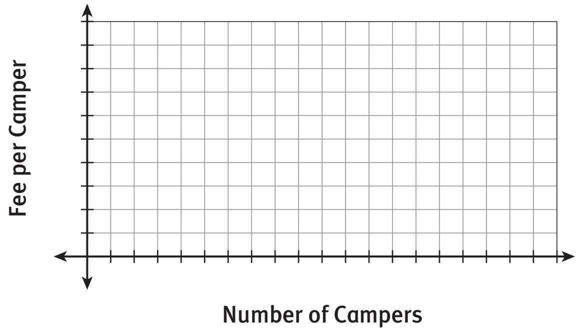
Use the patterns you observe in the table to write an algebraic expression in the last row when there are x campers.

My Notes

MATH TIP

You can use the values in your table to help you determine an appropriate scale for a graph.

c. Using an appropriate scale, make a graph showing the relationship between the fee per camper and the number of campers in attendance.



d. Write an algebraic rule for the fee per camper as a function of the number of campers in attendance.

2. Describe the features of the graph in Item 1c.

3. **Reason abstractly.** Based on your work so far, is there a minimum camper fee, not counting the cost of meals? If so, what is it? Explain.

Check Your Understanding

4. What is the fee per camper if there are 2000 campers?

5. What relationship exists between the number of campers and the fee per camper?

6. Describe the domain and range of the function in the context of this problem.

My Notes

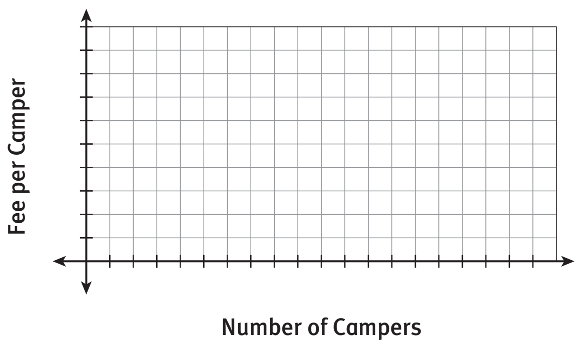
7. The function developed in Item 1 did not account for meals. Campers eat three meals per day at a cost of $3 per camper per meal. The committee must determine a function that includes the cost of meals when setting the fee per camper.

a. What will be the total cost for meals per camper each 5-day week?

b. Complete the table below to determine the fee per camper that will guarantee the camp does not lose money.

Number of Campers	Fixed Cost Plus the Cost of Meals	Fee per Camper
25		
50		
75		
100		
200		
500		
1000		
x		

c. Using an appropriate scale, make a graph showing the relationship between the fee per camper, including meals, and the number of campers.



d. Write an algebraic rule for the fee per camper, including meals, as a function of the number of children in attendance.

CONNECT TO AP

Describing the behavior of rational functions as they approach horizontal and vertical asymptotes provides an introduction to a more formal study of limits that will occur in calculus.

My Notes

8. Based on your work so far, is there a minimum camper's fee? If so, what is it? Explain your reasoning.

9. How does your answer to Item 8 differ from the one you gave for Item 3?

Check Your Understanding

10. What is the fee per camper if there are 2000 campers? Why is your answer different from what it was in Item 4?

11. Describe the domain and range of the function that includes the cost of meals in the context of this problem.

12. Describe the difference between the graphs in Items 1c and 7c.

LESSON 27-1 PRACTICE

A new start-up company is going to produce cell phone chargers. The fixed cost for the company is \$800 per day, which includes things such as rent, salaries, insurance, and equipment. The total daily cost, in dollars, to produce x chargers is $C(x) = 4x + 800$.

13. Evaluate and interpret $C(100)$ and $\frac{C(100)}{100}$.

14. Write an algebraic rule for finding the average cost per charger.

15. What relationship exists between the number of chargers and the average cost per charger?

16. Make a graph showing the relationship between the average cost per charger and the number of chargers produced. Include data points for producing 50, 100, 200, 400, 600, 800, and 1000 chargers.

17. **Make sense of problems.** The company has determined, based on warehouse space, equipment, and number of employees, that the maximum number of chargers they can produce in one day is 6000. Given this information, describe the domain and range of the average cost function.

My Notes

Learning Targets:

- Formulate rational equations that model real-world situations.
- Graph equations on coordinate axes.

SUGGESTED LEARNING STRATEGIES: Create Representations, Quickwrite, Self Revision/Peer Revision, Work Backward

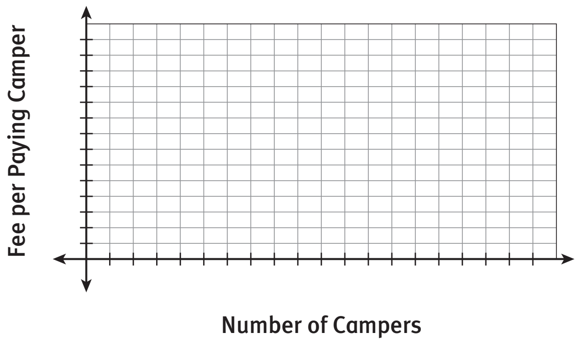
Summer camp is a rewarding experience for many young children and teenagers. It is not unusual for nonprofit organizations to award scholarships to students.

1. The finance committee from the nonprofit summer camp in Lesson 27-1 decides to award 30 scholarships to students who otherwise could not afford the camp. These scholarships include full use of the facilities and all meals at no charge.

a. To help account for the scholarships, complete the table below.

Number of Campers	Fixed Cost Plus the Cost of Meals	Number of Paying Campers	Fee per Paying Camper
50			
75			
100			
200			
500			
1000			
x			

b. Using an appropriate scale, make a graph showing the relationship between the fee per paying camper and the number of campers.



My Notes

c. Write an algebraic model for the fee per paying camper as a function of the number of campers in attendance.

d. What is the meaning of the numerator and the denominator in your model?

2. Based on your work so far, is there a minimum camper's fee? If so, what is it? Explain.

3. How does your answer to Item 2 differ from the one you gave for Item 8 in the previous lesson?

4. How does your graph in Item 1b compare to the one in Item 7c in the previous lesson?

Check Your Understanding

5. Compare the model you wrote in Item 1c to the one you wrote in Item 7d in the previous lesson. Why is the new denominator $x - 30$, rather than just x as before?

6. If the number of campers is 2000, what is the fee per paying camper? Why is this answer different from what it was in Item 10?

7. If the number of campers is 25, what is the fee per paying camper? What does your answer tell you about the limitations of this model?

8. Give the domain and range of the function for the fee per paying camper in the context of this problem. Write your answer using set notation.

Recall that as a nonprofit camp, the camp should not make any profit but still must cover its costs to stay in operation.

9. Last year the weekly camper fee was \$80. If the camp charges the same amount and grants 30 scholarships, what is the minimum number of paying campers that must attend so the camp does not lose money?

10. Express the number of campers as a function of the fee for each paying camper.

11. What is the relationship between the function in Item 1c and the function in Item 10?

Check Your Understanding

12. Explain how you could use the table of values you completed in Item 1a to check the reasonableness of your answer to Item 9.

13. Explain how you could use your answer to Item 10 to check the reasonableness of your answer to Item 9.

14. **Attend to precision.** Use the model $f(x) = \dfrac{12{,}000 + 45x}{x - 30}$ and a graphing calculator to find the minimum number of paying campers that must attend so the camp does not lose money if the weekly fee is \$100. Describe the process you used to find your answer.

My Notes

LESSON 27-2 PRACTICE

15. The summer camp can accommodate up to 300 campers, and market research indicates that campers do not want to pay more than $200 per week. Although the camp is nonprofit, it cannot afford to lose money. Write a proposal for setting the fee per camper. Be sure to include these items.

- the proposed fee
- the minimum number of campers needed to break even
- the maximum possible income for the proposed fee
- mathematics to support your reasoning

Model with mathematics. The population of grizzly bears in a remote area is modeled by the function $P(t) = \frac{200t - 120}{t + 0.5}$, where $t = 1$ represents the year 2001, $t = 2$ represents the year 2002, and so on. Use the model to answers Items 16–20.

16. Graph the grizzly bear population function.

17. Describe the features of the graph.

18. What are the domain and range of the function?

19. How many grizzly bears were there in 2005?

20. Predict the bear population in the year 2018.

My Notes

Learning Targets:

- Determine the horizontal and vertical asymptotes of a rational function.
- Graph a rational function on the coordinate plane.

SUGGESTED LEARNING STRATEGIES: Vocabulary Organizer, Interactive Word Wall, Create Representations, Note Taking

When using a function to model a situation like the fee per camper, you only use those values that make sense in the context of the situation. In Items 1–3, we will consider the ***rational function*** $f(x) = \frac{12{,}000 + 45x}{x - 30}$ over a broader range of values.

1. Graph the function on a graphing calculator, using the viewing window [−450, 450] by [−400, 400].

   a. Use your calculator to approximate the *x*- and *y*-intercepts.

   b. Find the exact values of the *x*- and *y*-intercepts, using the function. Show your work.

   c. Recall that division of a nonzero quantity by zero is *undefined*. Name the value(s) for which the function is not defined and explain how you determined the value(s).

   d. What is the domain of the function? Write your answer using set notation, interval notation, and inequalities.

   e. What is the range of the function? Write your answer using set notation, interval notation, and inequalities.

MATH TERMS

A **rational function** is a function that is the quotient of two polynomials. Its parent function is $f(x) = \frac{1}{x}$.

TECHNOLOGY TIP

Throughout this book, graphing calculator viewing window dimensions are given as [xmin, xmax] by [ymin, ymax].

My Notes

MATH TERMS

A **horizontal asymptote** is the line $y = a$ when the end behavior of a function approaches some constant a.

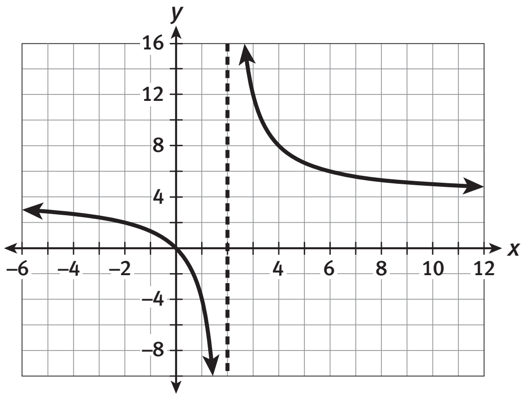
A **vertical asymptote** is the line $x = b$ if the absolute value of a function increases without bound as x approaches some number b.

TECHNOLOGY TIP

You can use the **Table** feature of your graphing calculator to examine end behavior.

If the values of a function f approach some number a as the absolute value of x becomes large without bound, the line $y = a$ is called a ***horizontal asymptote*** of f. If the absolute value of a function f increases without bound as x approaches some number b, then the line $x = b$ is a ***vertical asymptote*** of f.

2. The graph below shows the function $f(x) = \frac{4x}{x-2}$.



a. To examine the end behavior of the graph, use the function to complete the table. Round values to two decimal places.

x	−300	−100	−50	−20	−10	10	20	50	100	300
$f(x)$										

b. Write the equations of the horizontal and vertical asymptotes.

c. What do you notice about the equation of the vertical asymptote in relation to the denominator of the function?

Check Your Understanding

3. In Items 1d and 1e, you found the domain and range of the rational function $\frac{12{,}000 + 45x}{x - 30}$. Compare your answer to the domain and range you found in the last lesson (Item 8) when the same function was used to model the paying camper fees.

4. Use the information you gathered in Item 1 to write the equations of the vertical and horizontal asymptotes of $f(x) = \frac{12{,}000 + 45x}{x - 30}$.

My Notes

To graph a rational function, examine the equation and begin with key aspects, such as asymptotes and intercepts. Then fill in the "missing pieces" by plotting a few points around the asymptotes and intercepts.

5. Consider the function $R(x) = \dfrac{2x - 9}{x + 3}$.

a. Is $R(x)$ a rational function? Explain your reasoning.

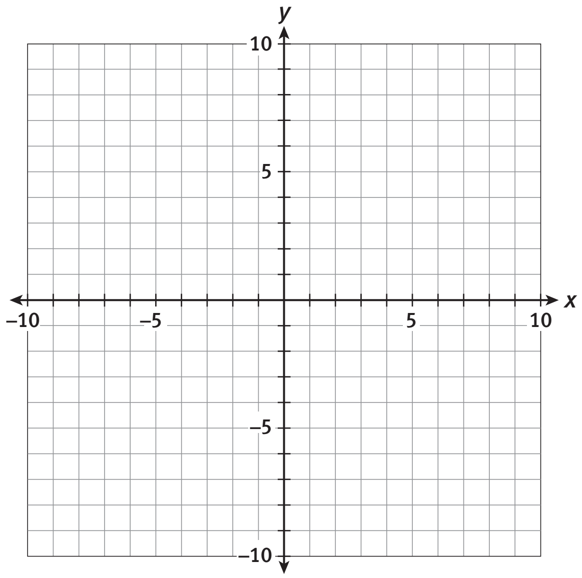
b. For what value of x is the function undefined? What does this tell you about the graph?

c. Examine the end behavior of the function by completing the table. Then write the equation of the horizontal asymptote.

x	−300	−100	−50	−20	−10	−5	10	20	50	100	300
$f(x)$											

d. Find the exact values of the x- and y-intercepts. Show your work.

e. Sketch the graph of $R(x)$ on the grid below. Include each piece of information you found in parts a–d. Use dotted lines to indicate asymptotes.



MATH TIP

The graph of a rational function will always approach (get very close to) its asymptotes.

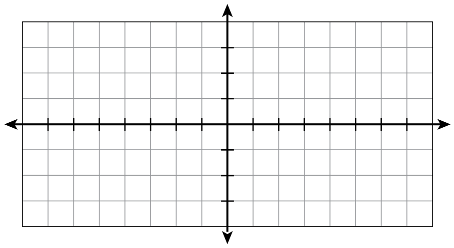
MATH TIP

Use points that you have already found, such as intercepts and the points in the end behavior table, to fill in your graph.

My Notes

Check Your Understanding

6. **Attend to precision.** Sketch the graph of the function in Item 1. Indicate the scale, label any intercepts, and include the horizontal and vertical asymptotes.



7. Given the function $f(x) = \frac{x+2}{x-3}$,
   - **a.** Identify any asymptotes of f.
   - **b.** Identify the x- and y-intercepts of f.
   - **c.** Sketch the graph of f.

LESSON 27-3 PRACTICE

8. Give the domain and range of the rational function in Item 5. Write your answers using set notation, interval notation, and inequalities.

9. Given the function $f(x) = \frac{2x+15}{x+3}$,
   - **a.** Identify any asymptotes of f.
   - **b.** Identify the x- and y-intercepts of f.
   - **c.** Sketch the graph of f.

10. Write an equation of a rational function that has a numerator of 1 and a vertical asymptote at $x = 5$.

11. **Use appropriate tools strategically.** Dwayne used a graphing calculator to graph the rational function $f(x) = \frac{2x-1}{3x+15}$.

    He was able to determine that the line $x = -5$ is a vertical asymptote of the graph by using the **Table** feature. Explain how the table shown here supports Dwayne's determination.

X	Y1
−7	2.5
−6	4.3333
−5	ERROR
−4	−3
−3	−1.167
−2	−.5556
−1	−.25

X=−1

ACTIVITY 27 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 27-1

Delaney decided to earn extra money by typing papers for people. Before she started her new endeavor, she had to purchase a printer. She paid $400 for the printer. She also determined that it would cost her approximately $0.05 per page in ink and paper.

1. Complete the table below to determine Delaney's average cost per page, including the cost of the printer.

Number of Pages	Cost per Page
20	
50	
100	
200	
400	
x	

2. Using an appropriate scale, make a graph showing the relationship between the cost per page and the number of pages typed.
3. Write an algebraic rule for the cost per page as a function of the number of pages Delaney types.
4. What relationship exists between the number of pages typed and the cost per page?
5. What is the cost per page if Delaney types 1000 pages?
6. Based on your work so far, is there a minimum cost per page? Is this realistic in the context of the problem? Explain your answer.

Lesson 27-2

Delaney wants to figure how much she should charge per page. She knows she needs to consider a variety of factors, such as what people are willing to pay, her costs, her time, and her profit. She has researched the issue and based on her research decides to charge $10 per page.

7. Complete the table below to determine Delaney's profit per page in relation to her cost per page.

Number of Pages	Cost per Page	Profit per Page
20		
50		
100		
200		
400		
x		

8. Explain the meaning of any negative values in the table.
9. Using an appropriate scale, make a graph showing the relationship between the profit per page and the number of pages typed.
10. The algebraic rule $p(x) = 10 - \frac{400 + 0.05x}{x}$ can be used to model Delaney's profit per page. Use algebra to rewrite the rule as a single rational function, $p(x) = \frac{q(x)}{r(x)}$.
11. Use your answer from Item 10 to find Delaney's profit per page if she types 1000 pages and if she types 2000 pages.
12. Predict where the horizontal asymptote is for the graph of $p(x)$. Explain your reasoning.

13. Which of the following is the equation of the vertical asymptote for $f(x) = \frac{6x-1}{x+1}$?

A. $x = 6$
B. $x = 1$
C. $x = \frac{1}{6}$
D. $x = -1$

14. Given the function $f(x) = \frac{2x-9}{x+3}$,

a. Identify any asymptotes of *f*.
b. Identify the *x*- and *y*-intercepts of *f*.
c. Sketch the graph of *f*.

15. Which of the following is the range of the function in Item 14?

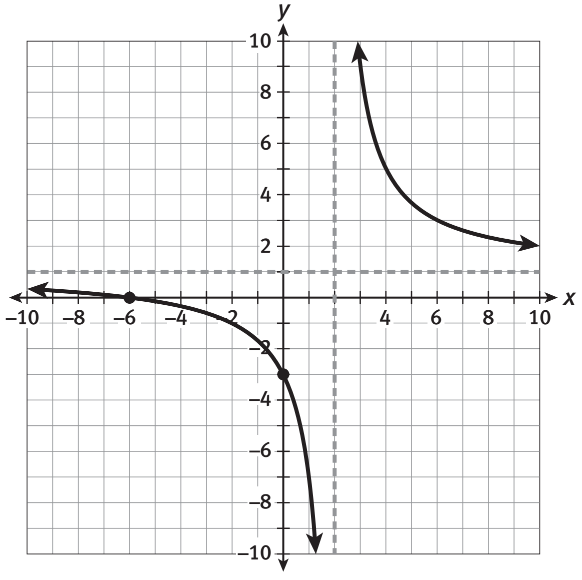
A. $(-\infty, -3) \cup (-3, \infty)$
B. $(-\infty, 2) \cup (2, \infty)$
C. $(-\infty, 3) \cup (3, \infty)$
D. $(-\infty, 4.5) \cup (4.5, \infty)$

16. Write an equation of a rational function that has a numerator of *x* and a vertical asymptote at $x = -4$.

17. What is the domain of the function you created in Item 16? Write your answer in interval notation.

MATHEMATICAL PRACTICES
Look For and Make Use of Structure

18. Consider the graph of a rational function.



a. What is the *x*-intercept of the graph?
b. Write a linear equation with a leading coefficient of 1 which has the same *x*-intercept as this graph.
c. What is the equation of the vertical asymptote of the graph?
d. A vertical asymptote is the result of a 0 in the denominator of a rational function. Use this fact to write a linear expression that could be the denominator of the rational function.
e. Write a rational function using your answers to parts b and d to help you determine the numerator and denominator.
f. What is the *y*-intercept of your function from part e? Does it match the *y*-intercept of the graph?
g. Graph your function from part e on your graphing calculator. Does it match the graph at the beginning of the problem? What do this and your answer from part f tell you about your function?

Inverse Variation and Rational Functions

ACTIVITY 28

Stream Survival

Lesson 28-1 Inverse Variation and Combined Variation

My Notes

Learning Targets:

- Create, solve, and graph an equation involving inverse variation.
- Solve an equation involving combined variation.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Create Representations, Look for a Pattern, Vocabulary Organizer, Work Backward

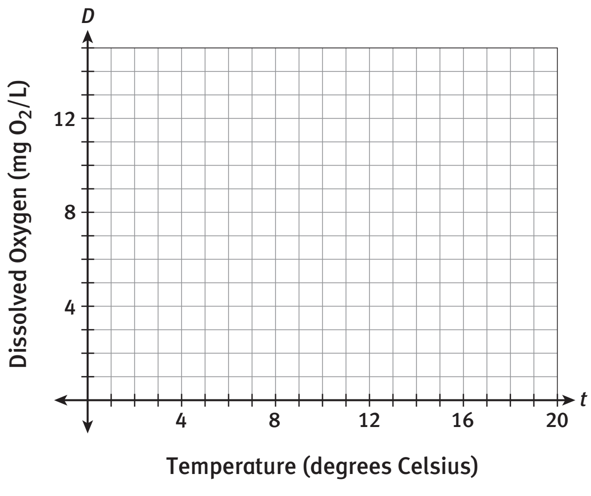
The amount of dissolved oxygen in a body of water decreases as the water temperature increases. Dissolved oxygen needs to be at sufficient levels to sustain the life of aquatic organisms such as fish. The table shows the temperature t and the corresponding amount of dissolved oxygen D in a stream that flows into Lake Superior on several dates from May to August.

CONNECT TO ECOLOGY

Fish and other aquatic organisms need oxygen to live, just as mammals do. Dissolved oxygen in the water passes through fish gills and then is transferred into the bloodstream. When dissolved oxygen levels in the water are too low, not enough oxygen will move into the bloodstream of the fish to maintain life.

Date	t (° Celsius)	D (mg O_2/L)	
May 1	11.5	10.6	
May 15	12.5	9.8	
June 1	13.0	9.5	
June 15	14.0	8.7	
July 1	14.5	8.5	
July 15	15.0	8.1	
Aug 1	16.5	7.4	

1. Graph the data above as a set of points on the axes.



2. Are these data linear? Explain why or why not.

3. Add a fourth column to the table, showing the product of t and D.

4. **Make use of structure.** What do you observe about the products of t and D that you recorded in the table?

My Notes

MATH TERMS

Inverse variation is written in the form $y = \frac{k}{x}$, where x and y are the variables that vary inversely.

The **constant of variation** is the constant k in the inverse variation equation. k is equal to the product of x and y.

CONNECT TO TECHNOLOGY

Use a graphing calculator to compute a power regression for the data and compare that equation to the one you wrote.

Inverse Variation **Equation**

When the product of two variable quantities x and y is constant, the two variables are said to vary inversely.

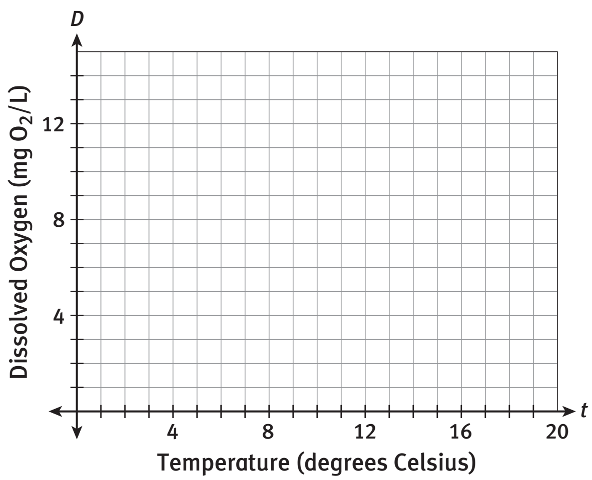
If $xy = k$ and $x \neq 0$, then $y = \frac{k}{x}$, where k is the ***constant of variation***.

Although the products of t and D from Item 3 are not constant, the products are close in value. When you use mathematics to model a real-world situation, the functions do not always give exact results.

5. If you use inverse variation to model the dissolved oxygen and temperature relationship, what value would you choose for k?

6. Write an inverse variation equation relating t and D that shows a constant product. Then solve the equation for D.

7. **Use appropriate tools strategically.** Use your calculator to make a scatter plot of the points (t, D) and graph the equation from Item 6 on the axes in Item 1.



8. How well does the model that you created fit the data?

9. When dissolved oxygen is less than 6 mg O_2/L, salmon are in danger. Use the model to find the maximum safe temperature for salmon.

My Notes

Check Your Understanding

Write and use an inverse variation equation to solve each problem.

10. y varies inversely as x. When x is 5, y is 10. Find y when x is 18.

11. The length of a rectangle varies inversely as its width. If the area is 40 in.^{2} and the width is 12.5 in., find the length of the rectangle.

12. Boyle's Law says that the volume of a gas in a closed container at constant temperature is inversely proportional to the pressure of the gas. Suppose 5 L of a gas are at a pressure of 2.0 atmospheres. What is the volume when the pressure is 3.0 atmospheres?

Another type of variation is ***combined variation***. Recall that two unknowns x and y vary *directly* if they are related by the equation $y = kx$ and *inversely* if $y = \frac{k}{x}$, where k, in both cases, is a nonzero constant.

Combined variation occurs when a variable varies directly with one variable and inversely with another variable. Combined variation is written in the form $y = \frac{kx}{z}$, where the constant of variation is k. y varies directly with x and inversely with z.

In Item 12, Boyle's Law results in an inverse variation as long as the temperature is constant. However, if both the pressure and the temperature vary, then the relationship between the volume, pressure, and temperature of a gas can be represented by a combined variation. This relationship is referred to as the combined gas law and states that the volume of a gas in a closed container varies directly with the temperature of the gas and inversely with the pressure of the gas.

MATH TERMS

Combined variation is a combination of direct and inverse variation, written in the form $y = \frac{kx}{z}$, where k is the constant of variation.

MATH TERMS

Joint variation is another form of variation and is written in the form $y = kxz$.

CONNECT TO CHEMISTRY

A common unit of pressure is the kilopascal, abbreviated kPa. A *torr* is a less common unit of pressure used in measuring partial vacuums. It is equal to approximately 133.32 pascals.

The kelvin, abbreviated K, is a unit of temperature often used in science. Unlike the more familiar Celsius and Fahrenheit scales, kelvin units are not referred to as "degrees." 300 K is about 27° C.

Example A

If 2 liters of a gas have a pressure of 1,500 torr and a temperature of 300 kelvins, what is the volume when the temperature is 600 kelvins and the pressure is 750 torr?

Step 1: Write a combined variation formula. $V = \frac{kT}{P}$

Step 2: Substitute $V = 2$, $T = 300$, and $P = 1500$ to find k. $2 = \frac{k(300)}{1500}$, $k = 10$

Step 3: Write the combined variation equation. $V = \frac{10T}{P}$

Step 4: Find V when $T = 600$ and $P = 750$. $V = \frac{10(600)}{750} = 8$

Solution: The volume is 8 liters.

My Notes

Try These A

Find k, and then write and use a combined variation equation to solve each problem.

a. y varies directly as x and inversely as z. When $x = 3$ and $z = 4$, $y = 9$. Find y when $x = 4$ and $z = 2$.

b. If 205 mL of a gas have a pressure of 30.8 kPa and a temperature of 451 kelvins, what is the volume when the temperature is reduced to 300 kelvins and the pressure is reduced to 100 kPa?

Check Your Understanding

13. The electrical resistance R of a wire varies directly as the length l of the wire and inversely as the square of the wire's diameter, d. This relationship can be written as the combined variation equation $R = \frac{kl}{d^2}$. If a wire of length 50 feet and diameter 4 millimeters has a resistance of 2 ohms, what is the resistance of a wire made of the same metal that is twice as long but has a diameter of 2 millimeters?

14. Write a combined variation equation to represent the following relationship: y varies directly as the square of x and inversely as the cube root of z.

LESSON 28-1 PRACTICE

15. Describe the relationship that exists between two variables that vary inversely.

16. The number of hours h it takes for ice to melt completely varies inversely as the temperature T, assuming the temperature is above freezing. If it takes 2 hours for a square inch of ice to melt at 65° F, how long does it take the ice to melt if the temperature is 50° F? Explain your reasoning.

17. Write a combined variation equation to represent the following relationship: y varies directly as the square root of x and inversely as three times z.

18. Given the relationship in Item 17, if y and z both equal 2 when x is 9, what is y when x is 36 and z is 8?

19. **Reason quantitatively.** Use the combined variation equation you found for the resistance R of the wire in Item 13 to determine which of the following wires (made of the same metal as in Item 13) has a greater resistance: a wire that is 100 feet long and 6 mm in diameter or a wire that is four times as long and twice as wide. Justify your answer.

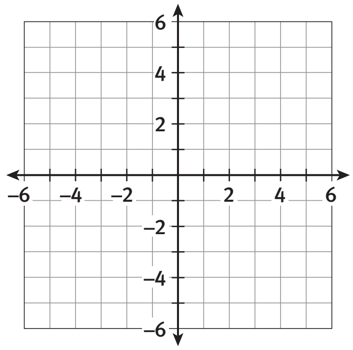
Learning Targets:

- Describe transformations of the parent function $f(x) = \frac{1}{x}$ and sketch the graphs.
- Identify the x-intercepts, y-intercepts, and asymptotes of transformations of the parent function $f(x) = \frac{1}{x}$.

SUGGESTED LEARNING STRATEGIES: Vocabulary Organizer, Create Representations, Look for a Pattern, Predict and Confirm, Note Taking

The rational function $f(x) = \frac{1}{x}$ is an example of an inverse variation equation whose constant of variation is 1.

1. Make a table of values and graph the parent rational function $f(x) = \frac{1}{x}$.



2. Construct viable arguments. Does your graph cross the y-axis? If so, give the y-intercept. If not, explain using algebra why it does not. Then answer the same questions about the x-axis.

3. Describe the key features of $f(x) = \frac{1}{x}$. Use appropriate mathematics vocabulary in your description.

My Notes

MATH TIP

The basic rational function is sometimes called the *reciprocal function*. It can be graphed easily by plotting the ordered pairs $\left(n, \frac{1}{n}\right)$.

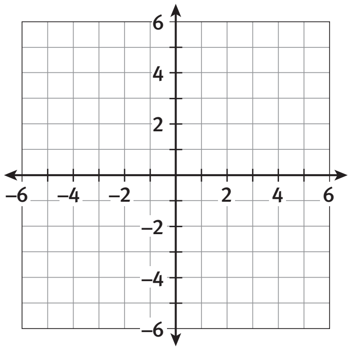
My Notes

MATH TIP

Given $y = f(x)$, the function $y = a \cdot f(x)$ represents a *vertical stretch or shrink* of the original function whose y-values have been multiplied by $a > 0$. You have a vertical stretch for $a > 1$ and a vertical shrink for $0 < a < 1$.

Functions like the one in the last lesson modeling dissolved oxygen and temperature are a *vertical stretch* of the parent function $f(x) = \frac{1}{x}$.

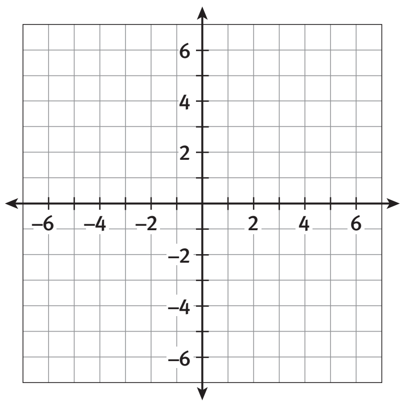
4. Enter the functions $f(x) = \frac{1}{x}$, $g(x) = \frac{2}{x}$, and $h(x) = \frac{5}{x}$ into your graphing calculator. Sketch the graphs on the axes below.



5. How do the y-values of g and h compare to those of the parent function?

6. Describe the similarities and the differences in the graphs of those three functions.

7. Sketch the parent function $f(x) = \frac{1}{x}$ and the graph of $k(x) = \frac{3}{x}$ on the same axes without using your graphing calculator.



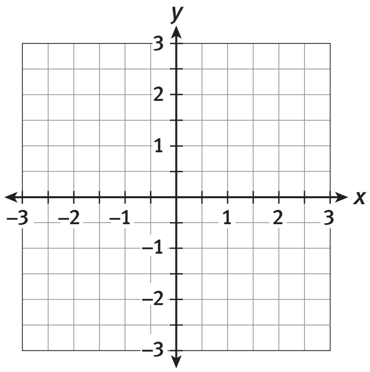
Lesson 28-2
Transformations of the Parent Rational Function

So far, we've sketched vertical stretches of the parent function $f(x)=\frac{1}{x}$.
Now let's examine transformations of the form $h(x)=\frac{1}{bx}$.

8. Use algebra to write $h(x)=\frac{1}{bx}$ as a product of a coefficient and the parent function $f(x)=\frac{1}{x}$.

9. In Item 1, you made a table of values to graph the parent function $f(x)=\frac{1}{x}$. Use your answer to Item 8 to describe how to change the y-coordinates in the table to graph the function $h(x)=\frac{1}{2x}$.

10. Graph the parent function $f(x)=\frac{1}{x}$ and the function $h(x)=\frac{1}{2x}$ on the same axes without using your graphing calculator.



11. Without using your calculator, predict what the graph of $f(x)=-\frac{1}{x}$ will look like.

12. In Item 1, you made a table of values to graph the parent function $f(x)=\frac{1}{x}$. Describe how to change the y-coordinates of the points on the graph of the parent function to graph $g(x)=-\frac{2}{x}$.

My Notes

MATH TIP

Given $y=f(x)$, the function $y=-f(x)$ represents a *vertical reflection* of the original function whose y-values have been multiplied by -1.

My Notes

Check Your Understanding

13. Sketch the graph of $k(x) = -\frac{2}{x}$.

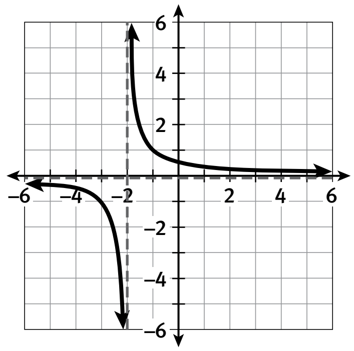
14. Where are the horizontal and vertical asymptotes on the graphs of $g(x) = \frac{a}{x}$, $h(x) = \frac{1}{bx}$, and $k(x) = -\frac{1}{x}$ in relation to the asymptotes of the parent function $f(x) = \frac{1}{x}$? Explain your reasoning.

MATH TIP

Given $y = f(x)$, the function $y = f(x + c)$ results in a *horizontal translation* of the original function and $y = f(x) + c$ results in a *vertical translation* of the original function.

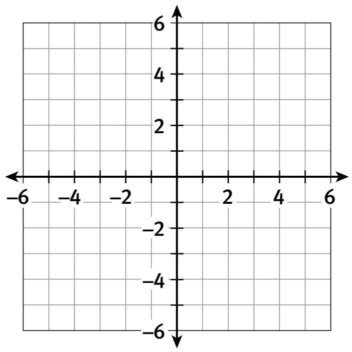
15. Sketch the graph of each function and then describe it as a transformation of the parent function $f(x) = \frac{1}{x}$. The first graph has been done for you.

a. $f(x) = \frac{1}{x+2}$



Transformation: The graph is translated 2 units to the left.

b. $f(x) = \frac{1}{x-2}$

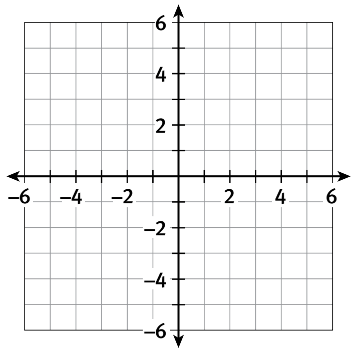


Transformation:

MATH TIP

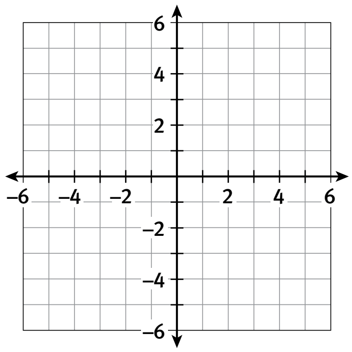
A good strategy for translating rational functions is to first move the asymptotes and then fill in key points.

c. $f(x) = \frac{1}{x} + 2$



Transformation:

d. $f(x) = \frac{1}{x} - 2$



Transformation:

Check Your Understanding

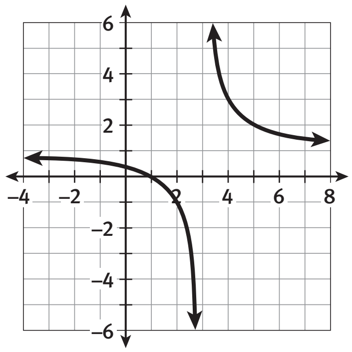
16. Describe each function as a transformation of $f(x)=\frac{1}{x}$.

a. $f(x)=\frac{1}{x+1}$ **b.** $f(x)=\frac{1}{x}-3$ **c.** $f(x)=\frac{1}{x-5}+3$

17. Make use of structure. Without graphing, identify which functions in Item 16 cross the x-axis, y-axis, or both axes. Justify your answers.

Example A

Describe the function $f(x)=\frac{2}{x-3}+1$ as a transformation of $f(x)=\frac{1}{x}$. Identify the x- and y-intercepts and the asymptotes. Sketch the graph.

Transformations	**Asymptotes**
• vertical stretch by a factor of 2 • horizontal translation 3 units to the right • vertical translation 1 unit up	$x=3$ $y=1$
Intercepts y-intercept: $f(0)=\frac{1}{3}$ x-intercept: Solve $f(x)=0$. $\frac{2}{x-3}+1=0$ $\frac{2}{x-3}=-1$ $2=-1(x-3)$ $x=1$	

Try These A

On a separate sheet of grid paper, describe each function as a transformation of $f(x)=\frac{1}{x}$. Identify the x- and y-intercepts and the asymptotes. Sketch the graph.

a. $f(x)=\frac{3}{x}+1$ **b.** $f(x)=-\frac{1}{x+1}-2$ **c.** $f(x)=3+\frac{4}{x-2}$

My Notes

Check Your Understanding

18. The rational function $R(x)$ is a transformation of the parent function $f(x) = \frac{1}{x}$. The parent function has been vertically shrunk by a factor of 3 and then translated down 5 units. What are the equations of the asymptotes of $R(x)$?

19. Describe the function $g(x) = \frac{4}{x+1} - 3$ as a transformation of $f(x) = \frac{1}{x}$. Identify the x- and y-intercepts and the asymptotes. Sketch the graph.

LESSON 28-2 PRACTICE

20. Write a function that is $f(x) = \frac{1}{x}$ translated 3 units down and 5 units to the right.

21. Describe the graph of $g(x) = \frac{1}{4x}$ as a transformation of the parent function $f(x) = \frac{1}{x}$.

22. Given the rational function $R(x) = \frac{-2}{x} + 3$, identify the x- and y-intercepts and the asymptotes. Then sketch the graph.

23. The parent function $f(x) = \frac{1}{x}$ is translated 4 units up and 7 units to the right. Without graphing, identify the asymptotes.

24. **Attend to precision.** The point (1, 1) lies on the graph of the parent function $f(x) = \frac{1}{x}$. Where does this point translate to on the graph of $g(x) = \frac{2}{x+1} - 5$?

ACTIVITY 28 PRACTICE

Write your answers on notebook paper.
Show your work.

Lesson 28-1

1. Given the inverse variation $y = \frac{10}{x}$, what is the constant of variation?
   A. $k = -10$
   B. $k = 1$
   C. $k = 10$
   D. $k = 100$
2. If y varies inversely as x, and $y = 8$ when $x = 40$, which equation models this situation?
   A. $y = \frac{5}{x}$ **B.** $y = \frac{32}{x}$
   C. $y = \frac{48}{x}$ **D.** $y = \frac{320}{x}$
3. Evan's video game scores vary inversely as the number of games played without rest. If he scores 1,000 points after playing two games in a row, how much will he score after playing five games in a row?
4. The variable y varies directly as x and inversely as z. When $x = 4$ and $z = 14$, $y = 2$. Write a combined variation equation and find y when $x = 6$ and z = 3.
5. Using the same relationship as in Item 4, find z when $x = 14$ and $y = 49$.

Lesson 28-2

6. Write a function that is $f(x) = \frac{1}{x}$ translated 2 units up and 6 units to the left.
7. Which function is a vertical translation and a vertical stretch of $f(x) = \frac{1}{x}$?
   A. $f(x) = \frac{x}{2}$ **B.** $f(x) = \frac{1}{x+2}$
   C. $f(x) = \frac{2}{x} + 1$ **D.** $f(x) = \frac{1}{2} + x$

Use $g(x) = \frac{2}{x+1} - 5$ to answer Items 8–10.

8. What is the vertical asymptote of g?
   A. $x = -5$
   B. $x = -1$
   C. $x = 1$
   D. $x = 2$
9. What is the vertical stretch of g?
   A. 2 **B.** 1
   C. -5 **D.** none
10. Identify the asymptotes of g.

11. The point $\left(8, \frac{1}{8}\right)$ lies on the graph of the parent function $f(x) = \frac{1}{x}$. Where does the image of this point lie on the graph of $h(x) = \frac{1}{3x}$?

12. Describe the graph of $g(x) = -\frac{1}{2x}$ as a transformation of the parent function $f(x) = \frac{1}{x}$.

For Items 13–15, identify the *x*- and *y*-intercepts and the asymptotes of the function. Then sketch the graph.

13. $f(x) = \frac{1}{x+2} - 2$

14. $g(x) = \frac{-2}{x} + 5$

15. $h(x) = \frac{1}{2x} - 4$

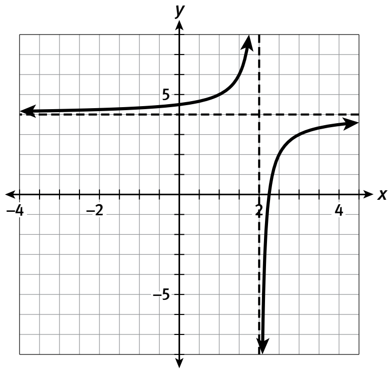
MATHEMATICAL PRACTICES
Model with Mathematics

16. Some satellites that orbit the earth travel in circular paths at extremely high velocities. The satellite's distance from the center of the earth is called the radius r of the orbit. The time T, in hours, that it takes for a satellite to complete one full orbit around the earth varies directly as the radius of the orbit and inversely as the satellite's velocity v in miles per hour. Suppose we know that it takes a satellite traveling at 11,340 mph 30 minutes to complete an orbit 900 miles above the earth.

a. Write a combined variation equation to model this relationship.

b. How many minutes will it take a satellite traveling at 12,600 mph to complete an orbit 1200 miles above the earth? Show your work.

1. **Model with mathematics.** KitKat Kondos makes kitty condos. They have $10,000 per week in fixed operating costs and each kitty condo costs $12 to make.
   a. Write a polynomial function that represents the cost of making x kitty condos.
   b. Create a table to use with KitKat condos. Label the columns Number of Condos, Total Cost, and Cost per Condo. Use 25, 50, 75, 100, 125, 150, 200, 250, 300, 400, and x for the number of condos.
   c. Write a rational function that represents the cost per condo of x kitty condos.
   d. Draw a graph showing the relationship between the number of condos and the cost per condo. Use your table to determine an appropriate scale.
   e. If the cost per condo was $13, how many condos did the company make?
2. The following are two real-world applications that can be modeled by variation equations. Give the type of variation for each equation and the value of k, the constant of variation.
   a. $s = \frac{d}{t}$, where s is the speed of a moving object that travels a distance d in a given amount of time s
   b. $f = \frac{5}{3l}$, where f is the force (pounds of pressure) needed to break a particular type of board l feet long
3. If y varies inversely as x and $y = 8$ when $x = \frac{1}{2}$, write and use an inverse variation equation to find y when $x = 2$.
4. The cost per person of renting a bus varies inversely with the number of people renting the bus. A particular bus can accommodate 56 people. If the bus is completely full, the cost per person is $15. Write and use an inverse variation equation to determine the cost per person if only 20 people rent the bus.
5. If y varies directly as x and inversely as z, and $y = 4$ when $x = 2$ and $z = 5$, write and use a combined variation equation to find y when $x = 6$ and $z = 3$.
6. Use what you have learned about transformations to write an equation for the rational function whose graph is shown here. Pay close attention to the graph's scales. Check your equation by finding the x- and y-intercepts.



7. Given the rational function $r(x) = -\frac{2}{x+3} - 1$.
   a. Describe $r(x)$ as a transformation of the parent function $f(x) = \frac{1}{x}$.
   b. Use what you know about transformations to identify the asymptotes of $r(x)$.
   c. Give the domain and range of $r(x)$. Write your answers in set notation and in interval notation.
   d. Find the x- and y-intercept of $r(x)$.
   e. Sketch the graph of $r(x)$. Include all of the key information from parts a to d and at least two additional points on the graph.

Rational Functions and Variation

A CONDO FOR MY CAT

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
	The solution demonstrates these characteristics:			
Mathematics Knowledge and Thinking (Items 3, 4, 5, 7b–d)	• Clear and accurate identification of key features of rational functions including domain and range, intercepts, and asymptotes and transformations of a parent function • Fluency in recognizing, writing, and evaluating inverse and combined variation equations	• Mostly accurate identification of key features of rational functions including domain and range, intercepts, and asymptotes and transformations of a parent function • Little difficulty in recognizing, writing, and evaluating inverse and combined variation equations	• Partially accurate identification of key features of rational functions including domain and range, intercepts, and asymptotes and transformations of a parent function • Some difficulty in recognizing, writing, and evaluating inverse and combined variation equations	• Incomplete or inaccurate identification of key features of rational functions including domain and range, intercepts, and asymptotes and transformations of a parent function • Significant difficulty in recognizing, writing, and evaluating inverse and combined variation equations
Problem Solving (Items 1, 4)	• An appropriate and efficient strategy that results in a correct answer	• A strategy that may include unnecessary steps but results in a correct answer	• A strategy that results in some incorrect answers	• No clear strategy when solving problems
Mathematical Modeling / Representations (Items 1, 2, 4, 6, 7e)	• Clear and accurate tables, graphs, equations, and rational functions to model a real-world scenario • Clearly and accurately recognize real-world scenarios modeled by inverse or combined variation equations • Fluency in applying transformations to graph rational functions and writing a rational function given its graph	• Mostly accurate tables, graphs, equations, and rational functions to model a real-world scenario • Mostly accurate recognition of real-world scenarios modeled by inverse or combined variation equations • Little difficulty in applying transformations to graph rational functions and writing a rational function given its graph	• Partially accurate tables, graphs, equations, and rational functions to model a real-world scenario • Partial recognition of real-world scenarios modeled by inverse or combined variation equations • Some difficulty in applying transformations to graph rational functions and writing a rational function given its graph	• Incomplete or mostly inaccurate tables, graphs, equations, and rational functions to model a real-world scenario • Difficulty in recognizing real-world scenarios modeled by inverse or combined variation equations • Significant difficulty in applying transformations to graph rational functions and writing a rational function given its graph
Reasoning and Communication (Items 2, 7)	• Precise use of appropriate math terms and language when identifying variation equations and features of rational functions • Clear and accurate description of transformations of a parent function	• Adequate use of math terms and language when identifying variation equations and features of rational functions • Adequate description of transformations of a parent function	• Misleading or confusing use of math terms and language when identifying variation equations and features of rational functions • Misleading or confusing description of transformations of a parent function	• Incomplete or mostly inaccurate use of appropriate math terms and language when identifying variation equations and features of rational functions • Incomplete or inadequate description of transformations of a parent function

Simplifying Rational Expressions

It's All Rational

Lesson 29-1 Multiplying and Dividing Rational Expressions

My Notes

Learning Targets:

- Simplify rational expressions.
- Multiply and divide rational expressions.

SUGGESTED LEARNING STRATEGIES: Interactive Word Wall, Vocabulary Organizer, Note Taking, Marking the Text, Simplify the Problem

Rational expressions can be simplified and combined, using the operations of addition, subtraction, multiplication, and division.

Writing rational expressions in simpler forms and combining them helps you to understand and graph rational functions and solve equations.

To simplify a rational expression, factor the numerator and denominator. Identify the restrictions on the variable x that make the denominator in the expression equal to zero. Then, divide out the common factors.

MATH TIP

The variable in a rational function must be restricted so that the denominator will not be equal to zero.

Example A

Simplify each expression.

a. $\dfrac{x^2+5x-14}{x^2-4}$

b. $\dfrac{2x^2+7x+3}{x^2+7x+12}$

Step 1: Identify the restrictions on x. Set the denominators equal to zero.

a. $x^2-4=0$
$(x+2)(x-2)=0$
$x+2=0$ or
$x-2=0$
$x=-2$ or
$x=2$

b. $x^2+7x+12=0$
$(x+3)(x+4)=0$
$x+3=0$ or $x+4=0$
$x=-3$ or $x=-4$

Step 2: Factor the numerators and denominators.

a. $\dfrac{x^2+5x-14}{x^2-4} = \dfrac{(x+7)(x-2)}{(x+2)(x-2)}$

b. $\dfrac{2x^2+7x+13}{x^2+7x+12} = \dfrac{(2x+1)(x+3)}{(x+4)(x+3)}$

Step 3: Divide out common factors.

a. $= \dfrac{(x+7)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}$
$= \dfrac{x+7}{x+2},\ x \neq 2, -2$

b. $= \dfrac{(2x+1)\cancel{(x+3)}}{(x+4)\cancel{(x+3)}}$
$= \dfrac{2x+1}{x+4},\ x \neq -3, -4$

MATH TIP

Make sure that you use the original rational expression when identifying restrictions on the variable.

My Notes

Try These A

Simplify. Identify any restrictions on x. Show your work.

a. $\dfrac{x^2 + 20x + 36}{x^3 - 4x}$

b. $\dfrac{x^2 - 2x - 15}{2x^2 + 3x - 9}$

c. $\dfrac{x^3 - 9x}{3 - x}$

Check Your Understanding

1. You are given a rational expression and told that the expression must be restricted so that $x \neq 3, -3$. What is the denominator of the expression you were given?
2. **Construct viable arguments.** Is it possible for a rational expression to have no restrictions? If so, give an example and explain why it has no restrictions. If not, explain why not.

To multiply rational expressions and express the product in lowest terms, factor the numerator and denominator of each expression. Then, divide out any common factors. Any restrictions on the value of x in the original expressions will also apply to the simplified expression.

Example B

Simplify the expression. Identify any restrictions on x.

Original expression $\dfrac{2x^2 - 8}{x^2 - 1} \cdot \dfrac{x^2 + 2x + 1}{x^3 - x^2 - 2x}$

Step 1: Factor the numerators and denominators. $\dfrac{2(x+2)(x-2)}{(x+1)(x-1)} \cdot \dfrac{(x+1)(x+1)}{x(x-2)(x+1)}$

Step 2: Divide out common factors. $\dfrac{2(x+2)\cancel{(x-2)}\cancel{(x+1)}\cancel{(x+1)}}{\cancel{(x+1)}(x-1)x\cancel{(x-2)}\cancel{(x+1)}}$

$\dfrac{2(x+2)}{x(x-1)}, x \neq 2, 1, 0, -1$

My Notes

To divide rational expressions, write division as multiplication and then finish simplifying the expression using the same steps as in Example B.

Example C

Simplify the expression. Identify any restrictions on x.

Original expression $\dfrac{x^2+5x+6}{x^2-4} \div \dfrac{5x+15}{3x^2-4x-4}$

Step 1: Write as multiplication. $\dfrac{x^2+5x+6}{x^2-4} \cdot \dfrac{3x^2-4x-4}{5x+15}$

Step 2: Factor the numerators and denominators. $\dfrac{(x+2)(x+3)}{(x+2)(x-2)} \cdot \dfrac{(3x+2)(x-2)}{5(x+3)}$

Step 3: Divide out common factors. $\dfrac{\cancel{(x+2)}\,\cancel{(x+3)}\,(3x+2)\,\cancel{(x-2)}}{\cancel{(x+2)}\,\cancel{(x-2)}\,(5)\,\cancel{(x+3)}}$

$\dfrac{3x+2}{5}, x \neq -\dfrac{2}{3} -2, 2, -3$

MATH TIP

When dividing numerical fractions, write as multiplication.

$$y = \frac{2x}{x+5}$$

If a, b, c, and d have any common factors, you can cancel them before you multiply.

$$\frac{4}{15} \div \frac{8}{3} = \frac{4}{15} \cdot \frac{3}{8} =$$

$$\frac{\cancel{4}}{\cancel{3} \cdot 5} \cdot \frac{\cancel{3}}{2 \cdot \cancel{4}} = \frac{1}{10}$$

Try These B–C

Perform the indicated operation. Identify any restrictions on x. Write your answers on notebook paper. Show your work.

a. $\dfrac{2x+4}{x^2-25} \cdot \dfrac{x^2-5x-50}{4x^2-16}$

b. $\dfrac{6x^2}{3x^2-27} \div \dfrac{2x+2}{x^2-2x-3}$

Check Your Understanding

3. Reason quantitatively. Explain the restrictions on x for the expressions in Try These B–C.

4. Multiply the expression. Identify any restrictions on x.

$$\frac{x^2+3x-4}{x^2-1} \cdot \frac{x^2-4x-5}{2x+8}$$

5. Divide the expression. Identify any restrictions on x.

$$\frac{x^2-25}{x^2+6x-7} \div \frac{x^2+13x+40}{x^2+7x-8}$$

My Notes

LESSON 29-1 PRACTICE

6. Simplify the expression. Identify any restrictions on x.

$$\frac{2x^2 - 5x - 12}{10x + 15}$$

7. Give a possible denominator for a rational expression that has restrictions $x \neq 4, -7$.

8. Attend to precision. Before graphing a rational function, you simplify the function and identify the restrictions on x. Explain what those restrictions tell you about the graph of the function.

9. Multiply the expression. Identify any restrictions on x.

$$\frac{4x - 4}{2x^2 - 9x - 5} \cdot \frac{x^2 - 5x}{x - 1}$$

10. Divide the expression. Identify any restrictions on x.

$$\frac{3x^2 - 3}{5x^2 + 9x + 4} \div \frac{21x - 21}{25x^2 - 16}$$

11. Critique the reasoning of others. Carmella simplified the following rational expression.

$$\frac{2x^2 + 7x + 3}{4x + 12} =$$

$$\frac{(2x + 1)(x + 3)}{4(x + 3)} =$$

$$\frac{(2x + 1)\cancel{(x + 3)}}{4\cancel{(x + 3)}} =$$

$$\frac{2x + 1}{4}$$

After simplifying the expression, she stated that there are no restrictions on x since the denominator of her answer is a constant and cannot possibly equal 0. Did Carmella make any mistakes in her problem? If so, explain and correct her mistake(s).

My Notes

Learning Targets:

- Add and subtract rational expressions.
- Simplify complex fractions.

SUGGESTED LEARNING STRATEGIES: Note Taking, Summarizing, Paraphrasing, Graphic Organizer, Simplify the Problem, Look for a Pattern

To add or subtract rational expressions with unlike denominators, find a common denominator. The easiest way to find the least common denominator is to factor the expressions. Then, the least common denominator is the product of each factor common to the expressions and any noncommon factors.

Example A

Find the least common denominator of $\frac{1}{x^2-3x-4}$ and $\frac{1}{x^2-16}$.

Step 1: Factor each denominator.

$x^2-3x-4=(x+1)(x-4)$
$x^2-16=(x+4)(x-4)$

Step 2: Identify common factors and factors not in common.

Factors in Common: $x-4$

Factors Not in Common: $x+4,\ x+1$

Step 3: Write the least common denominator.

$(x+4)(x+1)(x-4)$

Try These A

Find the least common denominator of $\frac{1}{x^2-9}$ and $\frac{1}{3x^2-9x}$.

Now you are ready to add and subtract rational expressions with different denominators.

My Notes

CONNECT TO AP

You will continue to use the skill of simplifying rational expressions in AP Calculus.

MATH TIP

When the denominators are the same, all you have to do is add or subtract the numerators as indicated by the operation.

Example B

Simplify the expression. Identify any restrictions on x.

Original expression $\frac{2}{x-2} - \frac{3}{x^2-2x} =$

Step 1: Factor the denominators. $\frac{2}{x-2} - \frac{3}{x(x-2)}$

Step 2: Find the least common denominator. $x(x-2)$

Step 3: Multiply the numerator and denominator of each term by the missing factor(s) of the least common denominator. $\frac{2(x)}{x(x-2)} - \frac{3}{x(x-2)} =$

Step 4: Subtract the like fractions to find the solution. $\frac{2x-3}{x(x-2)}, x \neq 2, 0$

Try These B

Simplify each expression. Identify any restrictions on x. Write your answers on notebook paper. Show your work.

a. $\frac{3}{x+1} - \frac{x}{x-1}$

b. $\frac{2}{x} - \frac{3}{x^2-3x}$

c. $\frac{2}{x^2-4} + \frac{x}{x^2+4x+4}$

d. $\frac{x}{x+2} + \frac{4}{x-3}$

e. $\frac{1}{x} + \frac{2x}{x^2-3}$

f. $\frac{2}{x^2-9} - \frac{3}{x^2}$

Check Your Understanding

1. **Reason quantitatively.** Why is it necessary to find a common denominator when adding rational expressions?
2. Simplify each expression. Identify any restrictions on x.

   a. $\frac{8}{x+5} + \frac{2x}{x+5} - \frac{x+3}{x+5}$

   b. $\frac{3x}{x^2-9} + \frac{1}{x+3}$
3. Find the least common denominator of $\frac{1}{5x+10}$ and $\frac{2}{x^2+4x+4}$.
4. Subtract: $\frac{2}{x^2-3x-4} - \frac{1}{x^2-1}$. Identify any restrictions on x.

Lesson 29-2
Adding and Subtracting Rational Expressions

You can simplify ***complex fractions*** if you treat them like a division problem. Simplify the numerator and denominator as much as possible, and then rewrite the problem using multiplication. Restricted values of x include any values that make any of the denominators in the original expression or the simplified expression equal to 0.

My Notes

MATH TERMS

A rational expression that contains rational expressions in its numerator and/or its denominator is called a **complex fraction**.

Example C

Simplify $\dfrac{1+\dfrac{1}{x+1}}{x-\dfrac{x}{x-1}}$. Identify any restrictions on x.

Original expression $\dfrac{1+\dfrac{1}{x+1}}{x-\dfrac{x}{x-1}} =$

Step 1: Simplify the numerator and denominator using their least common denominators. $\dfrac{\dfrac{x+1}{x+1}+\dfrac{1}{x+1}}{\dfrac{x(x-1)}{(x-1)}-\dfrac{x}{x-1}} =$

Step 2: Add or subtract fractions in the numerator and denominator. Combine like terms. $\dfrac{\dfrac{x+1+1}{x+1}}{\dfrac{x^2-x-x}{x-1}} = \dfrac{\dfrac{x+2}{x+1}}{\dfrac{x^2-2x}{x-1}} =$

Step 3: Write division as multiplication of the reciprocal. $\dfrac{x+2}{x+1} \cdot \dfrac{x-1}{x^2-2x} =$

Step 4: Factor and simplify if possible. $\dfrac{(x+2)(x-1)}{x(x+1)(x-2)}, x \neq 2, 1, 0, -1$

Try These C

Simplify. Identify any restrictions on x. Write your answers on notebook paper. Show your work.

a. $\dfrac{\dfrac{x^2-3x-4}{x^2-4}}{\dfrac{2x^2+2x}{x+2}}$

b. $\dfrac{\dfrac{x}{x+1}-\dfrac{1}{x-1}}{\dfrac{1}{x+1}+2}$

Check Your Understanding

5. Why is 0 a restricted value for x in Example C?

6. Attend to precision. Based on your work in this activity, do you think the set of rational expressions is closed under the operations of addition, subtraction, multiplication, and division by a nonzero rational expression? Explain.

My Notes

LESSON 29-2 PRACTICE

7. Simplify. Identify any restrictions on x.

$$\frac{3x}{x-2}+\frac{x}{x-2}-\frac{8}{x-2}$$

8. Find the least common denominator:

$$\frac{2}{x+5},\ \frac{x}{x^2+7x+10},\ \text{and}\ \frac{2x}{x^2-4}$$

9. Add the expressions. Identify any restrictions on x.

$$\frac{3}{x+2}+2$$

10. Simplify. Identify any restrictions on x.

$$\frac{1}{x+1}+\frac{x}{x-6}-\frac{5x-2}{x^2-5x-6}$$

11. Simplify. Identify any restrictions on x.

$$\frac{\frac{2x^2-10x-28}{x^2-4}}{\frac{x+4}{x^2-9x+14}}$$

12. Make sense of problems. Consider the complex rational expression below.

$$\frac{3-\frac{2}{x-4}}{5+\frac{4}{x-4}}$$

a. How is it different from the expression in Item 11?
b. What is the first step in simplifying this expression?
c. Simplify the expression. Identify any restrictions on x.

My Notes

Learning Targets:

- Identify the vertical asymptotes of rational functions by finding the domain values that make the functions undefined.
- Use the degrees of the numerator and denominator of rational functions to identify the horizontal asymptotes.

SUGGESTED LEARNING STRATEGIES: Vocabulary Organizer, Note Taking, Look for a Pattern, Marking the Text, Create Representations

In the graph of a rational function, a break in the graph often signals that a ***discontinuity*** has occurred. Algebraically, a discontinuity happens for values of x that cause the function to be undefined and are therefore not in the domain of the function.

Common factors that are divided out of a function during the simplification process produce *holes* in the graph, rather than *asymptotes*. These holes are called ***removable points of discontinuity***. The domain of a rational function does not include values of x where there are holes or vertical asymptotes.

MATH TIP

If $f(x) = \frac{p(x)}{q(x)}$, where p and q are polynomial functions in standard form with no common factors other than 1, then the function f has a *vertical asymptote* at each value of x for which $q(x) = 0$.

Example A

Identify any vertical asymptotes and any holes in the graph.

Step 1: Factor the numerator and denominator.

$$f(x) = \frac{x^2 - 4}{x^2 + 5x + 6}$$

$$f(x) = \frac{(x+2)(x-2)}{(x+2)(x+3)}$$

Step 2: Divide out the common factors.

$$f(x) = \frac{x-2}{x+3},\ x \neq -2$$

Step 3: Find the values that make the simplified denominator = 0.

$x + 3 = 0$ when $x = -3$
vertical asymptote $x = -3$

Step 4: Find other values that make the original denominator = 0.

$x + 2 = 0$ when $x = -2$
hole at $x = -2$

MATH TIP

To find the vertical asymptote of a graph, determine the values of the variables that make the function undefined when it is in simplest form.

Try These A

Identify any vertical asymptotes and any holes in the graph.

a. $f(x) = \frac{x^2 - x}{x^2 + 3x - 4}$

b. $f(x) = \frac{3 - x}{9 - x^2}$

TECHNOLOGY TIP

Use a graphing calculator to graph the function and visually see the breaks where the asymptotes are located. The calculator does not show holes in a graph, but there is still an error message in the calculator's Table feature for those values of x.

My Notes

Check Your Understanding

1. Give the domain of the rational functions in Example A and Try These A. Write your answers in set notation and interval notation.
2. **Make use of structure.** Use the factored form of $f(x)$ below to identify any vertical asymptotes and any holes in the graph of the function.

   $f(x) = \frac{2x(x+4)}{x(x+4)(x-6)}$

3. Give the domain of the rational function in Item 2. Write your answer in set notation and in interval notation.

MATH TIP

Since a horizontal asymptote describes end behavior, it is possible for a graph to cross the asymptote. There are algebraic techniques for determining if and when a graph crosses its horizontal asymptote.

A *horizontal asymptote* depends on the degrees of the numerator and denominator and describes the end behavior of a rational function.

- When the degrees are the same, the horizontal asymptote is the ratio of the leading coefficients.
- When the denominator degree is larger, the horizontal asymptote is equal to 0.
- When the numerator degree is larger, there is no horizontal asymptote.

Example B

Identify the horizontal asymptote, if any.

a. $f(x) = \frac{2+x}{x^2-1}$

numerator degree = 1

denominator degree = 2

$2 > 1$

horizontal asymptote: $y = 0$

b. $f(x) = \frac{2x+2}{x-1}$

numerator degree = 1

denominator degree = 1

leading coefficients: 2, 1

ratio of leading coefficients: 2

horizontal asymptote: $y = 2$

Try These B

Identify the horizontal asymptote, if any, of each function.

a. $f(x) = \frac{2-x}{x+4}$ **b.** $f(x) = \frac{x^2-1}{x+3}$ **c.** $\frac{x}{x^2-4}$

My Notes

Check Your Understanding

4. **Make sense of problems.** Give the range of the rational functions in Example B, part b, and Try These B, part a, given that neither of the graphs crosses its horizontal asymptote. Write your answers in set notation and interval notation.
5. Determine the degree and leading coefficient of the numerator and denominator of $f(x) = \frac{2x(x+4)}{x(x+4)(x-6)}$. Then give the equation of the horizontal asymptote, if any.

Now you are ready to use your knowledge of simplifying rational expressions to help you understand and graph rational functions.

To graph rational functions, follow these steps.

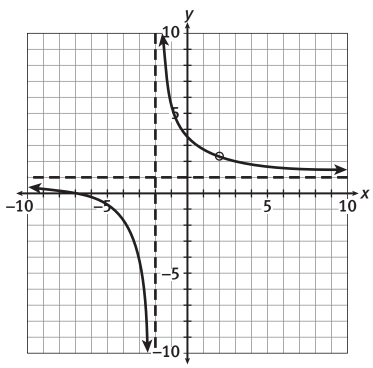
- Simplify the rational function.
- Express the numerator and denominator in factored form.
- Identify vertical asymptotes and holes.
- Identify x- and y-intercepts.
- Identify horizontal asymptote (end behavior).
- Make a sketch, using a graphing calculator as needed.

MATH TIP

Don't forget: If you divide out a common factor during the simplification process, there will be a hole in the graph where that factor equals zero.

Example C

Analyze and graph the rational function $f(x) = \frac{x^2 + 5x - 14}{x^2 - 4}$.

Simplify. $\frac{x^2+5x-14}{x^2-4} = \frac{(x+7)(x-2)}{(x+2)(x-2)} = \frac{(x+7)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}} = \frac{x+7}{x+2}$	**Identify vertical asymptotes and holes.** vertical asymptote is $x = -2$; hole at $x = 2$
	Identify intercepts. x-intercept: $x + 7 = 0$, so $x = -7$; y-intercept: $f(0) = \frac{0+7}{0+2} = 3.5$
Identify horizontal asymptote. numerator degree = 1; denominator degree = 1; leading coefficients: 1, 1; ratio of leading coefficients: 1; horizontal asymptote: $y = 1$	**Graph.** 

My Notes

Try These C

Analyze and graph the rational function. Show your work.

$$f(x)=\frac{x^2-4}{x^3-3x^2-10x}$$

Check Your Understanding

6. Give the domain of the function in Try These C using set notation and interval notation.
7. **Reason abstractly.** Use your analysis of the function in Try These C to explain how you know that the graph does indeed cross its horizontal asymptote. Then give the range of *f* using set notation and interval notation.

LESSON 29-3 PRACTICE

8. Identify any vertical asymptotes or holes of $f(x)=\frac{x^2-25}{x^2-2x-35}$.
9. Give the domain of the function in Item 8 using set notation and interval notation.
10. Identify any horizontal asymptotes of the function in Item 8.
11. Given that the graph of the function in Item 8 does not cross its horizontal asymptote, give the range of the function using set notation and interval notation.
12. **Make use of structure.** Use the steps demonstrated in Example C to analyze and graph the rational function $f(x)=\frac{x^2+2x}{x^2-x-6}$.

My Notes

Learning Targets:

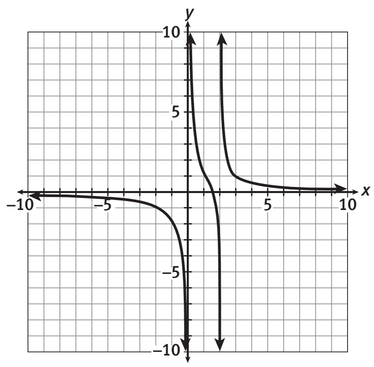
- Analyze and graph rational functions, identifying any asymptotes, intercepts, and holes.
- Analyze and graph rational functions representing real-world scenarios.

SUGGESTED LEARNING STRATEGIES: Graphic Organizer, Note Taking, Create Representations, Look for a Pattern, Marking the Text

To analyze and graph the sum or difference of two or more rational expressions, it is necessary to combine them into a single expression first. This makes it easier to identify the asymptotes and the intercepts.

Example A

Analyze and graph the rational function $f(x) = \frac{2}{x-2} - \frac{3}{x^2-2x}$.

Simplify. $\frac{2}{x-2} - \frac{3}{x^2-2x} =$ $\frac{2}{x-2} - \frac{3}{x(x-2)} =$ $\frac{2(x)}{x(x-2)} - \frac{3}{x(x-2)} =$ $\frac{2x-3}{x(x-2)}$	**Identify vertical asymptotes and holes.** $x = 0$ and $x - 2 = 0$, so vertical asymptotes are $x = 0$ and $x = 2$ no holes
	Identify intercepts. x-intercept: $2x - 3 = 0$, so $x = 1.5$ y-intercept: none, because $f(x)$ is undefined when $x = 0$
Identify horizontal asymptote. numerator degree = 1 denominator degree = 2 horizontal asymptote is $y = 0$	**Graph.** 

My Notes

Try These A

Analyze and graph the rational function. Write your answers on grid paper. Show your work.

$$f(x) = \frac{1}{x+1} - \frac{2}{x+3}$$

Check Your Understanding

1. Does the graph in Example A cross the horizontal asymptote? How do you know?
2. Give the domain and range of the rational function in Example A. Write your answers in set notation and interval notation.
3. Analyze and graph the rational function $f(x) = \frac{x}{x+1} + \frac{1}{x-1}$.

A common business application involving rational functions is finding average cost per unit. Before deciding to make a new product, businesses conduct a very thorough cost analysis, with the primary question being whether their anticipated cost per unit will be small enough that they can make a reasonable profit.

Example B

A recording studio has fixed costs of $18,000 to lay down tracks for a new album. This includes studio time, equipment, musicians, etc. It costs $1.20 to make each CD.

Write a linear function $C(x)$ giving the total cost of producing x CDs.	Add the fixed cost and the per-unit cost. $C(x) = 18{,}000 + 1.2x$
Write a rational function $A(x)$ giving the average cost per CD.	Divide by the number of units x. $A(x) = \frac{C(x)}{x} = \frac{18{,}000 + 1.2x}{x}$
Find any vertical asymptotes.	Set the denominator $= 0$: $x = 0$
Find the horizontal asymptote for $A(x)$.	ratio of leading coefficients: 1.2 horizontal asymptote: $y = 1.2$
What is the domain of $A(x)$ in the context of the problem?	The domain is the set of counting numbers: $\{1, 2, 3, \ldots\}$

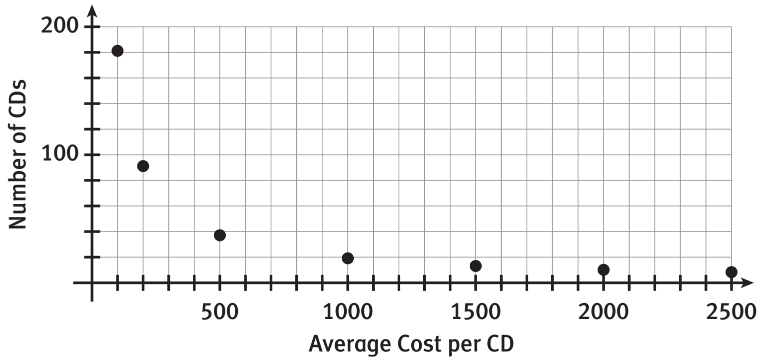
My Notes

Graph $A(x) = \frac{18,000 + 1.2x}{x}$ using the analysis of the function above.

Step 1: Complete a table of values to help decide on an appropriate scale. Remember that the domain is determined by the context of the problem.

x	100	200	500	1000	1500	2000	2500
$A(x)$	\$181.20	\$91.20	\$37.20	\$19.20	\$13.20	\$10.20	\$8.40

Step 2: Decide on a scale and label the axes. Include titles for the axes. Then graph the points from the table of values.



Step 3: Compare your graph to the analysis of the function.

The domain is discrete, so the data points are not connected by a curve. There appears to be a vertical asymptote at $x = 0$. As x gets larger, the functional values get closer to the x-axis, which corresponds to the horizontal asymptote at $y = 1.2$.

Check Your Understanding

4. Explain why the domain of $A(x)$ in Example B includes only the discrete values 1, 2, 3, … .

5. What is the average cost per CD if the studio produces 10,000 CDs? 15,000 CDs?

6. If the studio wants the average cost per CD to be \$2, how many CDs must they make? Justify your answer.

7. **Reason abstractly.** What meaning does the horizontal asymptote have in the context of the problem in Example B?

My Notes

LESSON 29-4 PRACTICE

8. Analyze and graph the rational function. Write your answer on grid paper. Show your work.

$f(x) = \frac{1}{x+2} + \frac{3}{x-3}$

9. Give the domain of the function in Item 8 using set notation and interval notation.

10. Give the range of the function using set notation and interval notation. Be sure to consider whether or not the function crosses its horizontal asymptote.

11. The graph of the function $f(x) = \frac{4}{x+3} - \frac{2x-18}{x^2-9}$ has one hole and one vertical asymptote. Determine the x-coordinate of the hole. Justify your answer.

12. Model with mathematics. A small printing company has accepted the job to print the yearbooks for all the high schools in their county. They have determined that the fixed costs for the project will be $14,000 and that it will cost them $12 to make each yearbook, which includes salaries, ink, paper, and binding. They have also informed the county that given the production schedule, the maximum number of yearbooks they can produce is 3000.

a. Write a rational function $A(x)$ giving the average cost per yearbook.

b. What is the domain of $A(x)$ in the context of the problem?

c. What is the average cost per yearbook if the schools order 1000 yearbooks?

d. Give the equation of the horizontal asymptote for $A(x)$. What meaning does the asymptote have in the context of the problem?

e. If together all the schools order 2000 yearbooks and the company charges them $40 for each one, what will the company's profit on this job be, assuming there were no unexpected costs? Explain your reasoning.

ACTIVITY 29 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 29-1

1. What are the restrictions on x in the rational expression $\frac{16-x^2}{4x+16}$?
   - **A.** none
   - **B.** $x \neq 0$
   - **C.** $x \neq -4$
   - **D.** $x \neq \pm 4$

2. Simplify $\frac{16-x^2}{4x+16}$. Identify any restrictions on x.

3. Give the restrictions on the expression $\frac{3}{x^2-5x-14}$ to determine where the graph of the rational function $f(x) = \frac{3}{x^2-5x-14}$ has vertical asymptotes.

4. Multiply $\frac{x^2-5x-6}{x^2-12x+36} \cdot \frac{x^2-36}{x^2-19x-20}$. Identify any restrictions on x.

5. Divide $\frac{4x+4}{x^2} \div \frac{x^2-1}{x^2-x}$. Identify any restrictions on x.

Lesson 29-2

6. Find the least common denominator of $\frac{1}{x-3}$, $\frac{x}{x^2-6x+9}$, and $\frac{2x}{x^2+7x-30}$.
   - **A.** $(x-3)$
   - **B.** $(x-3)(x+10)$
   - **C.** $(x-3)^2(x+10)$
   - **D.** $(x-3)(x+3)(x+10)$

7. Simplify $\frac{6}{x-6} + \frac{x}{x+6}$.
   - **A.** -1
   - **B.** $\frac{x+6}{x-6}$
   - **C.** $\frac{1}{x-6}$
   - **D.** $\frac{x^2+36}{(x-6)(x+6)}$

8. Simplify $\frac{2}{x+3} - \frac{x}{x-1}$.

9. Simplify. Identify any restrictions on x.

$$\frac{\frac{5}{x+6}}{\frac{10x}{x^2+3x-18}}$$

10. Simplify $\frac{\frac{1}{x-1}-\frac{1}{x}}{\frac{1}{x+1}-\frac{1}{x}}$. Identify any restrictions on x.

Lesson 29-3

For Items 11–13, identify any vertical asymptotes, holes, and horizontal asymptotes for the function.

11. $f(x) = \dfrac{2x+4}{x^2-4}$

12. $f(x) = \dfrac{x^2}{x^2+x-12}$

13. $f(x) = \dfrac{3x^2}{x^2+16}$

14. Give the domain of each of the functions in Items 11–13. Write your answers in set notation and in interval notation.

15. Use your graphing calculator to help you determine the range of the function in Item 13. Write your answer in set notation and in interval notation.

16. Analyze and graph $f(x) = \dfrac{x^2-36}{x^2-5x-6}$.

Lesson 29-4

For Items 17–20, use the rational function $f(x) = \dfrac{x-2}{x+5} + \dfrac{x^2+5x+6}{x^2+8x+15}$.

17. Identify any vertical asymptotes, holes, and horizontal asymptotes by first simplifying the function.

18. State the domain of the function. Write your answers in set notation and interval notation.

19. Find the x- and y-intercepts of the graph.

20. Sketch the graph using your answers to Items 17–19.

21. A company making surfboards has fixed costs of \$1600 per week. The cost to produce each surfboard is \$24.

a. Write a rational function $A(x)$ giving the average weekly cost per surfboard.

b. The maximum number of surfboards that the company can produce per week is 2000. Given this restriction, what is the domain of $A(x)$ in the context of the problem?

c. What is the average cost per surfboard if the company produces 1000 per week?

d. What does the average cost per surfboard approach as production increases?

e. What does your answer to part d tell you about the graph of the function?

MATHEMATICAL PRACTICES
Reason Abstractly and Quantitatively

Sometimes the graph of a rational function crosses its horizontal asymptote and sometimes it does not.

When the horizontal asymptote is $y = 0$, which is the x-axis, the graph will cross the asymptote wherever the simplified function has an x-intercept.

However, when the horizontal asymptote is not $y = 0$, you need to algebraically determine whether or not it crosses the asymptote. To do this, first find the horizontal asymptote and then set it equal to the original function. If you are able to find a solution for x, then you have found where the graph crosses the asymptote.

22. Determine whether the graphs of the functions in Items 11–13 cross their horizontal asymptotes. If so, tell where they cross. If not, explain how you determined that they do not cross.

Rational Equations and Inequalities

ACTIVITY 30

A Rational Pastime

Lesson 30-1 Solving Rational Equations

My Notes

Learning Targets:

- Solve rational equations, identifying any extraneous solutions.
- Create and solve rational equations that represent work problems.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Create Representations, Note Taking, Identify a Subtask, Guess and Check

Jesse pitches for the baseball team and wants to improve his batting average before the county all-stars are selected. To date, he has 10 hits out of 40 times at bat.

1. **Make sense of problems.** Batting average is the ratio of hits to at-bats. Write a ratio that represents Jesse's current batting average for this season and express the ratio in decimal form.

CONNECT TO MEASUREMENT

When a ratio is formed by two quantities with different units, it is also called a *rate*. Batting average is a rate, and even though we call it an average, it does not represent the mean of a set of numbers.

Jesse wants to improve his batting average to at least 0.320. If he gets a hit every time he bats, then his new batting average would be $\frac{10+x}{40+x}$, where x is the number of future hits in as many times at bat.

2. Write an equation to determine how many consecutive hits he needs to bat 0.320.

To solve equations like the one you wrote in Item 2, multiply by an expression that eliminates all the denominators.

Example A

Solve $\frac{x^2-4}{x+1}=x+5$

Original equation, undefined at $x=-1$ $\qquad \frac{x^2-4}{x+1}=x+5$

Step 1: Multiply both sides by $(x + 1)$ to cancel the denominator. $\qquad (x+1)\left(\frac{x^2-4}{x+1}\right)=(x+5)(x+1)$

Step 2: Solve for x.

$$x^2-4=x^2+6x+5$$
$$-4=6x+5$$
$$6x=-9$$
$$x=-1.5$$

Step 3: Check to see if the original equation is undefined at the solution.

MATH TIP

When checking your solutions, substitute the solution into the original equation.

My Notes

3. Solve the equation you wrote in Item 2 to find the number of consecutive hits that Jesse needs to increase his batting average.

Example B

Solve $\frac{2}{x} - \frac{1}{x+2} = \frac{3}{x}$.

Original equation, undefined at $x = 0$ and $x = -2$

$$\frac{2}{x} - \frac{1}{x+2} = \frac{3}{x}$$

Step 1: Multiply both sides by $x(x + 2)$ to cancel the denominators.

$$x(x+2)\left(\frac{2}{x} - \frac{1}{x+2}\right) = \left(\frac{3}{x}\right)x(x+2)$$

Step 2: Solve for x.

$$2(x + 2) - 1(x) = 3(x + 2)$$
$$2x + 4 - x = 3x + 6$$
$$x + 4 = 3x + 6$$
$$-2x = 2$$
$$x = -1$$

Step 3: Check to see if the original equation is undefined at the solution.

Try These A–B

Solve each equation and check your solution.

a. $\frac{x+4}{x+5} = \frac{3}{5}$

b. $\frac{2x}{x+2} - \frac{x}{x-1} = 1$

When solving a rational equation, it is possible to introduce an *extraneous solution*. The extraneous solution is not valid in the original equation, although it satisfies the polynomial equation that results when you multiply by the simplest common denominator.

4. Solve the equation. Identify any extraneous solutions. $\frac{1}{x} - \frac{2x}{x+2} = \frac{x-6}{x(x+2)}$.

My Notes

Check Your Understanding

Solve each equation. Identify any extraneous solutions. Show your work.

5. $\frac{x}{x-1} = \frac{1}{x-1} + \frac{2}{x}$

6. $\frac{1}{x} - \frac{x-1}{x^2+x} = \frac{x-1}{x+1}$

7. Construct viable arguments. Consider the equations in Try These A–B and Items 4–6. Compare the equations that did not have extraneous solutions to those that did. Make a conjecture about when a rational equation is likely to have an extraneous solution and what the extraneous solution might be.

Jesse's coach requires the team to help prepare the baseball diamond at school. Jesse and Cody, working together, can clean up the infield in 2 hours. If Jesse worked alone, it would take him 5 hours. To figure out how long it would take Cody to prepare the infield by himself, you must consider the portion of the job that can be completed in 1 hour.

8. If it takes Jesse 5 hours to complete the job, what fraction could he complete in 1 hour, assuming he works at an even pace?

9. If it takes Cody *t* hours to complete the job, what fraction could he complete in 1 hour, assuming he works at an even pace?

10. Jesse and Cody, working together, can clean up the infield in 2 hours. What fraction of the job can they complete in 1 hour if they work together?

11. Model with mathematics. Now write an equation using the verbal model below.

Jesse's work in 1 hour	+	Cody's work in 1 hour	=	Together work in 1 hour

12. Solve the equation you wrote in Item 11 to determine how long it would take Cody to complete the job if he worked alone.

My Notes

13. Use your answer to Item 12 to determine what fraction of the job Cody can complete in 1 hour.

14. Garrett has cleaned up the infield on his own before, and it took him 4 hours. How long will it take all three boys, working together, to prepare the infield for a game? Show your work.

Check Your Understanding

15. Reason quantitatively. Describe how you might check the reasonableness of your answers to Items 12 and 14.

16. Working alone, Christine can put together a 500-piece puzzle in 4 hours. Christine's mom takes 2 hours to put together the same puzzle. Write and solve an equation to determine how long it will take to put the puzzle together if Christine and her mom work as a team.

LESSON 30-1 PRACTICE

For Items 18–21, solve each equation. Identify any extraneous solutions.

17. $\frac{3}{x+2}-\frac{1}{5x}=\frac{1}{x}$

18. $\frac{4}{x}+7=\frac{2}{3x}$

19. $\frac{3}{x+1}+\frac{2}{x-4}=\frac{4x-11}{x^2-3x-4}$

20. $\frac{2}{x}-\frac{4}{x+1}=3$

21. Working together, Aaron and Rosa can mow the lawn around the local library in 3 hours. Working alone, Aaron can mow the lawn in 5 hours. Write and solve an equation to determine how long it would take Rosa to mow the lawn by herself.

22. Reason abstractly. Working together, Pipe 1 and Pipe 2 can fill a tank in 12 hours. Pipe 1 takes twice as long as Pipe 2 to fill the tank. Write and solve an equation to determine how long it would take for Pipe 2, working by itself, to fill the tank.

My Notes

Learning Targets:

- Solve rational inequalities by graphing.
- Solve rational inequalities by finding the sign of the inequality on either side of the numerator and denominator zeros.

SUGGESTED LEARNING STRATEGIES: Create Representations, Identify a Subtask, Look for a Pattern, Graphic Organizer, Note Taking

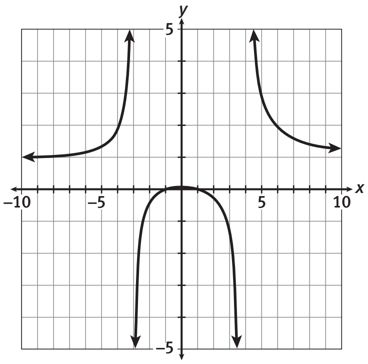
The rational inequality shown below can be solved graphically or numerically.

$$\frac{x^2-1}{x^2-x-12}<0$$

1. First, factor the left side of the inequality and determine the zeros and the values of x that are not in the domain of the function.

2. The graph of the left side of the inequality is shown below. The table shows the x- and y-coordinates and the sign of y.

x	y	sign
−5	1.333	+
−4	1.875	+
−3	undefined	
−2	−0.5	−
−1	0	
0	0.083	+
1	0	
2	−0.3	−
3	−1.333	−
4	undefined	
5	3	+



a. Identify the intervals of x where the graph is below the x-axis.

b. Attend to precision. Look back to the original inequality. Why would the intervals of x where the graph is below the x-axis be the solutions to the inequality?

My Notes

MATH TIP

To use this method, the inequality must be set up so that one side is a single rational expression and the other side is 0. If it is not, first transform the inequality into that form.

You can solve rational inequalities without using tables and graphs.

Solving Rational Inequalities

- Write the inequality in factored form.
- Identify the zeros of the numerator and the zeros of the denominator. (Note that the zeros of the denominator are the values where the rational function is not defined.)
- Pick one test value for x that falls between each of the zeros.
- Evaluate the left-hand side of the inequality at these values to test the sign of the inequality in each interval and determine the solution.
- State the solution intervals, and graph them on the number line.

Example A

Solve the inequality $\frac{x^2-1}{x^2-2x-8} \leq 0$

Factor:

$$\frac{(x+1)(x-1)}{(x-4)(x+2)} \leq 0$$

Zeros of the numerator at $x = 1$ and -1
Zeros of the denominator (where function is undefined) at $x = -2$ and 4

The zeros, in order from least to greatest, are: $-2, -1, 1, 4$
Pick and test one value in each interval: $-3, -1.5, 0, 2$, and 5

For $x = -3$:

$$\frac{(-3+1)(-3-1)}{(-3-4)(-3+2)} = \frac{(-2)(-4)}{(-7)(-1)} > 0$$

For $x = -1.5$:

$$\frac{(-1.5+1)(-1.5-1)}{(-1.5-4)(-1.5+2)} = \frac{(-0.5)(-2.5)}{(-5.5)(0.5)} < 0$$

MATH TIP

You do not need to evaluate the inequality completely. Simply plug in the test values and figure out the sign of each factor and then the overall sign of the rational expression.

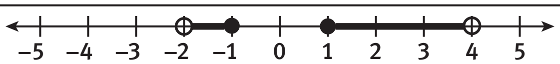
Continue this process and record the results in a table.

interval	$x < -2$	$-2 < x \leq -1$	$-1 \leq x \leq 1$	$1 \leq x < 4$	$x > 4$
test value	-3	-1.5	0	2	5
sign	$+$	$-$	$+$	$-$	$+$

The solution is the intervals of x where the inequality is less than or equal to 0 (recall the "≤ 0" in the original inequality). Therefore, x-values of the numerator zeros are included in the solution. x-values of the denominator are restricted and are not included in the solution.

Solution intervals: $-2 < x \leq -1$ or $1 \leq x < 4$

Graph the solution on a number line.



My Notes

Try These A

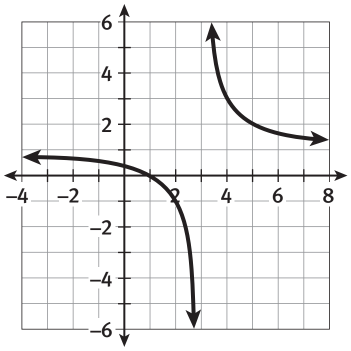
Solve each inequality algebraically or graphically.

a. $\frac{x^2-5x-6}{x^2-4x+3}\geq 0$

b. $\frac{1}{x}-\frac{2}{x+2}<0$

Check Your Understanding

3. Use the graph of $f(x)$ below to solve the inequality $f(x)\geq 0$.



4. Solve the inequality algebraically or graphically.

$$\frac{x^2+2x-15}{x-1}<0$$

A company making a new type of calculator has startup costs of $24,000. The cost to produce each calculator is $4.

5. **Model with mathematics.** Write an average cost function to represent the average cost per calculator.

6. The company's goal is to keep the average cost per calculator less than $9. Write an inequality to represent this scenario.

7. Will the average cost per calculator be less than $9 if the company makes 4000 calculators? Justify your answer.

8. Solve the inequality you wrote in Item 6 to determine the minimum number of calculators the company must make to meet their goal.

My Notes

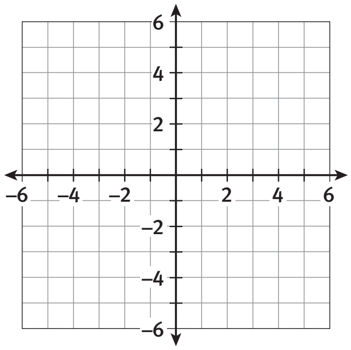
Check Your Understanding

9. The company making the calculators finds that they are able to lower their per-calculator production cost from \$4 to \$3 by using a different material to make the casing. Given this change, what is the minimum number of calculators they must produce to keep the average cost below \$9?

10. Make sense of problems. The company decides to sell the calculators for \$20 each. The function $P(x) = 20 - \frac{24{,}000 + 3x}{x}$ represents the average profit per calculator. The company's goal is to make a profit of at least \$15 per calculator. Write and solve an inequality to determine the minimum number of calculators they must produce to meet their goal.

LESSON 30-2 PRACTICE

11. Use the graph of $f(x) = \frac{3}{x} + 1$ below to identify the intervals of x where the graph is below the line $y = 1$.

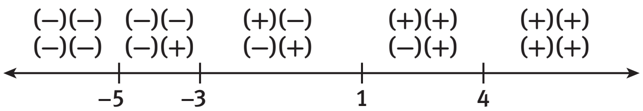


12. Use the graph to solve the inequality $\frac{3}{x} + 1 < 1$. Explain your reasoning.

13. Use the graph in Item 11 to solve the inequality $\frac{3}{x} + 1 < 0$.

14. Solve the inequality $\frac{x^2 + 5x - 14}{x^2 - 4} \geq 0$ algebraically or graphically.

15. Make use of structure. Ashton decided to combine the ideas of using a sign table and using a number line to solve an inequality. He made what he called the "sign line" below to solve the inequality $\frac{(x+3)(x-1)}{(x-4)(x+5)} < 0$.



Given that Ashton's sign line is correct, what is the solution to the inequality? Explain your reasoning.

ACTIVITY 30 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 30-1

1. Solve $\frac{x+3}{x-6}=\frac{2}{5}$.
A. $x=-1$
B. $x=-9$
C. $x=9$
D. no solution

2. Solve $\frac{2x}{x-1}+\frac{x-3}{x-1}=2$.
A. $x=1$
B. $x=0.5$
C. $x=1, 0.5$
D. no solution

Solve each equation in Items 3–6. Identify any extraneous solutions.

3. $\frac{x+3}{2}=\frac{5}{x}$

4. $\frac{2}{x}+\frac{3}{x}=\frac{5}{x+1}$

5. $\frac{x-3}{x-1}-\frac{2}{x+1}=\frac{x-5}{x^2-1}$

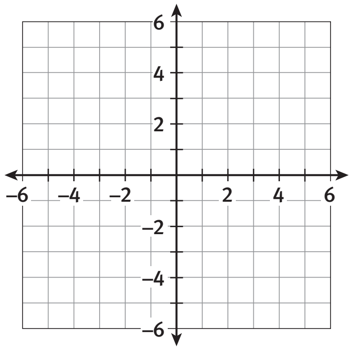
6. $\frac{1}{x-3}=\frac{x}{9-3x}$

7. Raj, Ebony, and Jed paint houses during the summer. Raj takes 5 hours to paint a room by himself, while it takes Ebony 4 hours and Jed 3 hours. How long will it take them if they all work together?

8. When Joe and Jon work together, they can wire a room in 8 hours. Working alone, Joe needs 12 hours to wire a room. How long would it take Jon, working alone, to wire a room?

Lesson 30-2

9. Use the graph below to determine which answer choice is the solution to the inequality $-\frac{1}{x+1}-2<0$.



A. $x<-1$
B. $x<-1.5$
C. $-1.5<x<-1$ or $x>-1$
D. $x<-1.5$ or $x>-1$

10. Use transformations to sketch the graph of $f(x)=-\frac{1}{x}$. Then solve the inequality $-\frac{1}{x}<0$.

For Items 11–14, solve each inequality algebraically or graphically.

11. $\frac{x^2-9}{x^2-4x-5}<0$

12. $\frac{1}{x+1}\geq\frac{2}{x-1}$

13. $\frac{x+7}{x^2-25}\leq 0$

14. $\frac{x^2-3x-18}{x^2-x-2}>0$

15. The prom committee is planning this year's prom. They have fixed costs of $2200 for music, decorations, and renting the ballroom. It will also cost $25 per person for the catered dinner.

a. Write a rational function that represents the average cost per person.

b. The prom committee's goal is to keep the average cost per person under $40. Write and solve an inequality to determine the minimum number of tickets the prom committee must sell to meet their goal.

c. The school principal reminds the prom committee that they must set aside 10 free tickets for the chaperones. The new average cost function is $A(x)=\frac{2200+25x}{x-10}$. How many more tickets must the committee sell to still keep the average cost per person under $40? Justify your answer.

MATHEMATICAL PRACTICES
Use Appropriate Tools Strategically

16. To solve rational inequalities using the method described in Lesson 30-2 Example A, the inequality must be set up so that one side is a single rational expression and the other side is 0. This often involves complex algebraic manipulation. You can use a graphing calculator as an alternate method to solve a rational inequality.

Consider the inequality $\frac{1}{x+5}+\frac{x}{x-3}<2$.

On your graphing calculator, enter the left side of the inequality as y_1 and the right side as y_2. Be sure to use parentheses around the denominator of each expression.

Graph the equations. Set your window as follows:

Xmin $=-15$

Xmax $=15$

Ymin $=-5$

Ymax $=5$

a. Draw a sketch of the graph you see on your calculator.

b. Use the calculator's **Intersect** feature to find the point(s) of intersection between y_1 and y_2. Add the point(s) to your sketch.

c. Explain how the point(s) of intersection help you solve the inequality.

d. Shade the portions of the graph that represent the solution to the inequality.

e. Solve the inequality.

Rational Expressions, Equations, and Inequalities

WORK IT OUT!

1. Perform the indicated operation. Simplify your answer if necessary. Identify any restrictions on x.

a. $\dfrac{2x-8}{x^2+5x-36} \bullet \dfrac{x^2+14x+45}{4x-12}$ **b.** $\dfrac{1}{x}+\dfrac{x}{2x+4}-\dfrac{2}{x^2+2x}$

c. $\dfrac{2-\dfrac{1}{x}}{4-\dfrac{1}{x^2}}$

2. For the rational function $f(x)=\dfrac{x^2+x}{x^2-x-2}$, give each of the following.

a. vertical asymptotes and holes
b. horizontal asymptotes
c. x- and y-intercepts
d. a sketch of the graph

3. **Make sense of problems.** A local contractor has three handymen that he always hires for remodeling bathrooms. Wayne is a very fast and competent worker who earns $24 per hour. Dashawn and Allen are not as experienced, take longer on projects, and earn $15 per hour.

The contractor is working on a house that has two equally sized bathrooms. From past experience, he knows that Wayne can finish a bathroom in 10 hours working alone, Dashawn can finish one in 14 hours, and Allen can finish one in 15 hours. The contractor wants to pair up two of the handymen in such a way that he minimizes his payroll. This means that one of the handymen will be working alone on his bathroom. When the team of two finishes, they will move to another job rather than helping finish the other bathroom.

a. Make a conjecture about which combination will minimize payroll: having Wayne work alone or having Wayne work with one of the slower handymen.
b. Develop a plan for determining which combination will minimize the contractor's payroll for this project. Be sure to include these items:
- the time it takes to complete each bathroom for each of the three possible combinations,
- the total payroll for each combination,
- which combination minimizes the payroll, and
- mathematics to support your reasoning.

4. Solve the equation $\dfrac{1}{x+3}+\dfrac{2}{x}=\dfrac{-3}{x^2+3x}$. State any extraneous solutions.

5. Solve the inequality $\dfrac{x^2+4x-12}{x^2-x-20}\le 0$ algebraically or graphically.

Rational Expressions, Equations, and Inequalities

WORK IT OUT!

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
	The solution demonstrates these characteristics:			
Mathematics Knowledge and Thinking (Items 1–5)	• Effective understanding and accuracy in simplifying rational expressions • Clear and accurate identification of key features of the graph of a rational function • Fluency in solving rational equations and inequalities	• Usually correct simplification of rational expressions • Mostly accurate identification of key features of the graph of a rational function • Little difficulty in solving rational equations and inequalities	• Difficulty when simplifying rational expressions • Partially accurate identification of key features of the graph of a rational function • Some difficulty in solving rational equations and inequalities	• Incomplete or mostly inaccurate simplification of rational expressions • Incomplete or inaccurate identification of key features of the graph of a rational function • Significant difficulty in solving rational equations and inequalities
Problem Solving (Item 3)	• An appropriate and efficient strategy that results in a correct answer	• A strategy that may include unnecessary steps but results in a correct answer	• A strategy that results in some incorrect answers	• No clear strategy when solving problems
Mathematical Modeling / Representations (Items 1d, 3)	• Effective understanding of how to graph a rational function • Fluency in developing, applying, and interpreting a model of a real-world scenario	• Largely correct understanding of how to graph a rational function • Little difficulty in developing, applying, and interpreting a model of a real-world scenario	• Partially correct understanding of how to graph a rational function • Some difficulty in developing, applying, and interpreting a model of a real-world scenario	• Inaccurate or incomplete understanding of how to graph a rational function • Significant difficulty in developing, applying, and interpreting a model of a real-world scenario
Reasoning and Communication (Item 3)	• Precise use of appropriate math terms and language to make and verify conjectures • Clear and accurate description of a plan to model a real-world scenario	• Adequate use of math terms and language when making and verifying conjectures • Adequate description of a plan to model a real-world scenario	• Misleading or confusing use of math terms and language when making and verifying conjectures • Misleading or confusing description of a plan to model a real-world scenario	• Incomplete or mostly inaccurate use of appropriate math terms and language when making and verifying conjectures • Incomplete or inadequate description of a plan to model a real-world scenario