Series, Exponential and Logarithmic Functions

4

Unit Overview

In this unit, you will study arithmetic and geometric sequences and series and their applications. You will also study exponential functions and investigate logarithmic functions and equations.

Key Terms

As you study this unit, add these and other terms to your math notebook. Include in your notes your prior knowledge of each word, as well as your experiences in using the word in different mathematical examples. If needed, ask for help in pronouncing new words and add information on pronunciation to your math notebook. It is important that you learn new terms and use them correctly in your class discussions and in your problem solutions.

Math Terms

- sequence
- arithmetic sequence
- common difference
- recursive formula
- explicit formula
- series
- partial sum
- sigma notation
- geometric sequence
- common ratio
- geometric series
- finite series
- infinite series
- sum of the infinite geometric series
- exponential function
- exponential decay factor
- exponential growth factor
- asymptote
- logarithm
- common logarithm
- logarithmic function
- natural logarithm
- Change of Base Formula
- exponential equation
- compound interest
- logarithmic equation
- extraneous solution

ESSENTIAL QUESTIONS

? How are functions that grow at a constant rate distinguished from those that do not grow at a constant rate?

? How are logarithmic and exponential equations used to model real-world problems?

EMBEDDED ASSESSMENTS

This unit has three embedded assessments, following Activities 20, 22, and 24. By completing these embedded assessments, you will demonstrate your understanding of arithmetic and geometric sequences and series, as well as exponential and logarithmic functions and equations.

UNIT 4

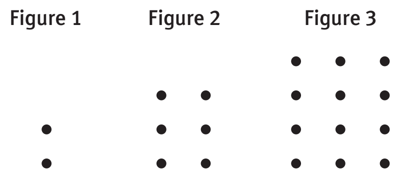
Getting Ready

Write your answers on notebook paper.
Show your work.

1. Describe the pattern displayed by 1, 2, 5, 10, 17, ...

2. Give the next three terms of the sequence 0, -2, 1, -3, ...

3. Draw Figure 4, using the pattern below. Then explain how you would create any figure in the pattern.



4. Simplify each expression.

a. $\left(\frac{6x^2}{y^3}\right)^2$

b. $(2a^2b)(3b^3)$

c. $\frac{10a^{12}b^6}{5a^3b^{-2}}$

5. Evaluate the expression.

$$\frac{3^{327}}{3^{323}}$$

6. Express the product in scientific notation.

$$(2.9 \times 10^3)(3 \times 10^2)$$

7. Solve the equation for x.

$$19 = -8x + 35$$

8. Write a function $C(t)$ to represent the cost of a taxicab ride, where the charge includes a fee of \$2.50 plus \$0.50 for each tenth of a mile t. Then give the slope and y-intercept of the graph of the function.

Arithmetic Sequences and Series

Arithmetic Alkanes

Lesson 19-1 Arithmetic Sequences

My Notes

Learning Targets:

- Determine whether a given sequence is arithmetic.
- Find the common difference of an arithmetic sequence.
- Write an expression for an arithmetic sequence, and calculate the *n*th term.

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Create Representations, Look for a Pattern, Summarizing, Paraphrasing, Vocabulary Organizer

Hydrocarbons are the simplest organic compounds, containing only carbon and hydrogen atoms. Hydrocarbons that contain only one pair of electrons between two atoms are called alkanes. Alkanes are valuable as clean fuels because they burn to form water and carbon dioxide. The number of carbon and hydrogen atoms in a molecule of the first six alkanes is shown in the table below.

Alkane	Carbon Atoms	Hydrogen Atoms
methane	1	4
ethane	2	6
propane	3	8
butane	4	10
pentane	5	12
hexane	6	14

1. **Model with mathematics.** Graph the data in the table. Write a function *f*, where $f(n)$ is the number of hydrogen atoms in an alkane with *n* carbon atoms. Describe the domain of the function.

MATH TERMS

A **sequence** is an ordered list of items.

Any function where the domain is a set of positive consecutive integers forms a ***sequence***. The values in the range of the function are the *terms* of the sequence. When naming a term in a sequence, subscripts are used rather than traditional function notation. For example, the first term in a sequence would be called a_1 rather than $f(1)$.

Consider the sequence {4, 6, 8, 10, 12, 14} formed by the number of hydrogen atoms in the first six alkanes.

2. What is a_1? What is a_3?

3. Find the differences $a_2 - a_1$, $a_3 - a_2$, $a_4 - a_3$, $a_5 - a_4$, and $a_6 - a_5$.

WRITING MATH

If the fourth term in a sequence is 10, then $a_4 = 10$.

Sequences may have a finite or an infinite number of terms and are sometimes written in braces { }.

Sequences like the one above are called *arithmetic sequences*. An ***arithmetic sequence*** is a sequence in which the difference of consecutive terms is a constant. The constant difference is called the ***common difference*** and is usually represented by *d*.

My Notes

4. Use a_n and a_{n+1} to write a general expression for the common difference d.

5. Determine whether the numbers of carbon atoms in the first six alkanes {1, 2, 3, 4, 5, 6} form an arithmetic sequence. Explain why or why not.

Check Your Understanding

Determine whether each sequence is arithmetic. If the sequence is arithmetic, state the common difference.

6. 3, 8, 13, 18, 23, . . .

7. 1, 2, 4, 8, 16, . . .

8. Find the missing terms in the arithmetic sequence 19, 28, ______, ______, 55, ______.

MATH TIP

In a sequence, a_{n+1} is the term that follows a_n.

9. Write a formula for a_{n+1} in Item 4.

10. What information is needed to find a_{n+1} using this formula?

Finding the value of a_{n+1} in the formula you wrote in Item 9 requires knowing the value of the previous term. Such a formula is called a ***recursive formula***, which is used to determine a term of a sequence using one or more of the preceding terms.

The terms in an arithmetic sequence can also be written as the sum of the first term and a multiple of the common difference. Such a formula is called an ***explicit formula*** because it can be used to calculate any term in the sequence as long as the first term is known.

11. Complete the blanks for the sequence {4, 6, 8, 10, 12, 14, . . .} formed by the number of hydrogen atoms.

$a_1 =$ ______ $d =$ ______

$a_2 = 4 +$ ______ $\cdot 2 = 6$

$a_3 = 4 +$ ______ $\cdot 2 = 8$

$a_4 = 4 +$ ______ $\cdot 2 =$

$a_5 = 4 +$ ______ $\cdot 2 =$

$a_6 = 4 +$ ______ $\cdot 2 =$

$a_{10} = 4 +$ ______ $\cdot 2 =$

My Notes

12. Write a general expression a_n in terms of n for finding the number of hydrogen atoms in an alkane molecule with n carbon atoms.

13. Use the expression you wrote in Item 12 to find the number of hydrogen atoms in decane, the alkane with 10 carbon atoms. Show your work.

14. Find the number of carbon atoms in a molecule of an alkane with 38 hydrogen atoms.

15. **Model with mathematics.** Use a_1, d, and n to write an explicit formula for a_n, the nth term of any arithmetic sequence.

16. Use the formula from Item 15 to find the specified term in each arithmetic sequence.

a. Find the 40th term when $a_1 = 6$ and $d = 3$.

b. Find the 30th term of the arithmetic sequence 37, 33, 29, 25, ...

Example A

Hope is sending invitations for a party. The cost of the invitations is \$5.00, and postage for each is \$0.45. Write an expression for the cost of mailing the invitations in terms of the number of invitations mailed. Then calculate the cost of mailing 16 invitations.

Step 1: Identify a_1 and d.
The cost to mail the first invitation is equal to the cost of the invitations and the postage for that one invitation.
$a_1 = 5.00 + 0.45 = 5.45.$
The postage per invitation is the common difference, $d = 0.45$.

Step 2: Use the information from Step 1 to write a general expression for a_n.
If n equals the number of invitations mailed, then the expression for the cost of mailing n invitations is:
$a_n = a_1 + (n - 1)d$
$a_n = 5.45 + (n - 1)(0.45)$
$a_n = 5.45 + 0.45n - 0.45$
$a_n = 5.00 + 0.45n$

Step 3: Use the general expression to evaluate a_{16}.
The cost of mailing 16 invitations is found by solving for $n = 16$.
$a_{16} = 5.00 + 0.45(16) = 5.00 + 7.20 = 12.20.$

Try These A

Write an expression for the nth term of the arithmetic sequence, and then find the term.

a. Find the 50th term when $a_1 = 7$ and $d = -2$.

b. Find the 28th term of the arithmetic sequence 3, 7, 11, 15, 19, ...

c. Which term in the arithmetic sequence 15, 18, 21, 24, . . . is equal to 72?

My Notes

Check Your Understanding

17. Show that the expressions for a_n in Item 12 and $f(n)$ in Item 1 are equivalent.

18. Find the 14th term for the sequence defined below.

term	1	2	3	4
value	1.7	1.3	0.9	0.5

19. Determine which term in the sequence in Item 18 has the value −1.1.

20. **Express regularity in repeated reasoning.** Shontelle used both the explicit and recursive formulas to calculate the fourth term in a sequence where $a_1 = 7$ and $d = 5$. She wrote the following:

Explicit:
$a_n = a_1 + (n-1)d$
$a_4 = 7 + (4-1)5$
$a_4 = 7 + 3 \times 5$

Recursive:
$a_n = a_{n-1} + d$
$a_4 = a_3 + 5$
$a_4 = (a_2 + 5) + 5$
$a_4 = ((a_1 + 5) + 5) + 5$
$a_4 = ((7 + 5) + 5) + 5$

Explain why Shontelle can substitute $(a_2 + 5)$ for a_3 and $(a_1 + 5)$ for a_2. Compare the result that Shontelle found when using the recursive formula with the result of the explicit formula. What does this tell you about the formulas?

LESSON 19-1 PRACTICE

For Items 21–23, determine whether each sequence is arithmetic. If the sequence is arithmetic, then

a. state the common difference.
b. use the explicit formula to write a general expression for a_n in terms of n.
c. use the recursive formula to write a general expression for a_n in terms of a_{n-1}.

21. 1, 1, 2, 3, 5, 8, . . .

22. 20, 17, 14, 11, 8, . . .

23. 3, 7, 11, . . .

24. A sequence is defined by $a_1 = 13$, $a_n = 5 + a_{n-1}$. Write the first five terms in the sequence.

25. **Make sense of problems.** Find the first term.

n	3	4	5	6
a_n	$\frac{7}{8}$	$\frac{3}{4}$	$\frac{5}{8}$	$\frac{1}{2}$

CONNECT TO HISTORY

Item 21 is a famous sequence known as the Fibonacci sequence. Find out more about this interesting sequence. You can find its pattern in beehives, pinecones, and flowers.

My Notes

Learning Targets:

- Write a formula for the nth partial sum of an arithmetic series.
- Calculate partial sums of an arithmetic series.

SUGGESTED LEARNING STRATEGIES: Look for a Pattern, Think-Pair-Share, Create Representations

A ***series*** is the sum of the terms in a sequence. The sum of the first n terms of a series is the nth ***partial sum*** of the series and is denoted by S_n.

1. Consider the arithmetic sequence {4, 6, 8, 10, 12, 14, 16, 18}.

a. Find S_4.

b. Find S_5.

c. Find S_8.

d. How does $a_1 + a_8$ compare to $a_2 + a_7$, $a_3 + a_6$, and $a_4 + a_5$?

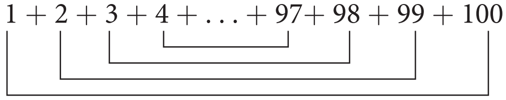
e. **Make use of structure.** Explain how to find S_8 using the value of $a_1 + a_8$.

2. Consider the arithmetic series $1 + 2 + 3 + \ldots + 98 + 99 + 100$.

a. How many terms are in this series?

b. If all the terms in this series are paired as shown below, how many pairs will there be?

$$1 + 2 + 3 + 4 + \ldots + 97 + 98 + 99 + 100$$



c. What is the sum of each pair of numbers?

d. **Construct viable arguments.** Find the sum of the series. Explain how you arrived at the sum.

CONNECT TO HISTORY

A story is often told that in the 1780s, a German schoolmaster decided to keep his students quiet by having them find the sum of the first 100 integers. One young pupil was able to name the sum immediately. This young man, Carl Friedrich Gauss, would become one of the world's most famous mathematicians. He reportedly used the method in Item 2 to find the sum, using mental math.

My Notes

3. Consider the arithmetic series $a_1 + a_2 + a_3 + \ldots + a_{n-2} + a_{n-1} + a_n$.

$$a_1 + a_2 + a_3 + \ldots + a_{n-2} + a_{n-1} + a_n$$

a. Write an expression for the number of pairs of terms in this series.

b. Write a formula for S_n, the partial sum of the arithmetic series.

4. Use the formula from Item 3b to find each partial sum of the arithmetic sequence {4, 6, 8, 10, 12, 14, 16, 18}. Compare your results to your answers in Item 1.

a. S_4

b. S_5

c. S_8

My Notes

5. A second form of the formula for finding the partial sum of an arithmetic series is $S_n = \frac{n}{2}[2a_1 + (n-1)d]$. Derive this formula, starting with the formula from Item 3b of this lesson and the nth term formula, $a_n = a_1 + (n - 1)d$, from Item 15 of the previous lesson.

6. Use the formula $S_n = \frac{n}{2}[2a_1 + (n-1)d]$ to find the indicated partial sum of each arithmetic series. Show your work.

a. $3 + 8 + 13 + 18 + \ldots; S_{20}$

b. $-2 - 4 - 6 - 8 - \ldots; S_{18}$

Example A

Find the partial sum S_{10} of the arithmetic series with $a_1 = -3$, $d = 4$.

Step 1: Find a_{10}.
The terms are $-3, 1, 5, 9, \ldots$.
$a_1 = -3$
$a_{10} = a_1 + (n - 1)d = -3 + (10 - 1)(4) = -3 + (9)(4) = -3 + 36 = 33$

Step 2: Substitute for n, a_1, and a_{10} in the formula. Simplify.
$S_{10} = \frac{n}{2}(a_1 + a_n) = \frac{10}{2}(-3 + 33) = 5(30) = 150$
Or use the formula $S_n = \frac{n}{2}[2a_1 + (n-1)d]$:
$S_{10} = \frac{10}{2}[2(-3) + (10 - 1)\,4] = 5[-6 + 36] = 150$

Try These A

Find the indicated sum of each arithmetic series. Show your work.

a. Find S_8 for the arithmetic series with $a_1 = 5$ and $a_8 = 40$.

b. $12 + 18 + 24 + 30 + \ldots; S_{10}$

c. $30 + 20 + 10 + 0 + \ldots; S_{25}$

My Notes

Check Your Understanding

7. Explain what each term of the equation $S_6 = 3(12 + 37) = 147$ means in terms of n and a_n.

8. Find each term of the arithmetic series in Item 7, and then verify the given sum.

9. When would the formula $S_n = \frac{n}{2}[2a_1 + (n-1)d]$ be preferred to the formula $S_n = \frac{n}{2}(a_1 + a_n)$?

LESSON 19-2 PRACTICE

10. Find the partial sum S_{10} of the arithmetic series with $a_1 = 4$, $d = 5$.

11. Find the partial sum S_{12} of the arithmetic series $26 + 24 + 22 + 20 + \ldots$.

12. Find the sum of the first 10 terms of an arithmetic sequence with an eighth term of 8.2 and a common difference of 0.4.

13. **Model with mathematics.** An auditorium has 12 seats in the first row, 15 in the second row, and 18 in the third row. If this pattern continues, what is the total number of seats for the first eight rows?

My Notes

Learning Targets:

- Identify the index, lower and upper limits, and general term in sigma notation.
- Express the sum of a series using sigma notation.
- Find the sum of a series written in sigma notation.

SUGGESTED LEARNING STRATEGIES: Look for a Pattern, Think-Pair-Share, Create Representations

In the Binomial Theorem activity in Unit 3, you were introduced to a shorthand notation called ***sigma notation*** (Σ). It is used to express the sum of a series.

The expression $\sum_{n=1}^{4}(2n+5)$ is read "the sum from $n = 1$ to $n = 4$ of $2n + 5$."

To expand the series to show the terms of the series, substitute 1, 2, 3, and 4 into the expression for the general term. To find the sum of the series, add the terms.

$$\sum_{n=1}^{4}(2n+5) = (2 \cdot 1 + 5) + (2 \cdot 2 + 5) + (2 \cdot 3 + 5) + (2 \cdot 4 + 5)$$
$$= 7 + 9 + 11 + 13 = 40$$

MATH TIP

upper limit of summation

$$\sum_{n=1}^{4}(2n+5)$$

general term

index of summation

lower limit of summation

MATH TIP

To find the first term in a series written in sigma notation, substitute the value of the lower limit into the expression for the general term.

To find subsequent terms, substitute consecutive integers that follow the lower limit, stopping at the upper limit.

Example A

Evaluate $\sum_{j=1}^{6}(2j-3)$.

Step 1: The values of j are 1, 2, 3, 4, 5, and 6. Write a sum with six addends, one for each value of the variable.
$= [2(1) - 3] + [2(2) - 3] + [2(3) - 3] + [2(4) - 3] + [2(5) - 3] + [2(6) - 3]$

Step 2: Evaluate each expression.
$= -1 + 1 + 3 + 5 + 7 + 9$

Step 3: Simplify.
$= 24$

Try These A

a. **Use appropriate tools strategically.** Write the terms in the series $\sum_{n=1}^{8}(3n-2)$. Then find the indicated sum.

b. Write the sum of the first 10 terms of $80 + 75 + 70 + 65 + \ldots$ using sigma notation.

My Notes

Check Your Understanding

Summarize the following formulas for an arithmetic series.

1. common difference $d =$ ____________

2. nth term $a_n =$ ____________

3. sum of first n terms $S_n =$ ____________
or
$S_n =$ ____________

LESSON 19-3 PRACTICE

Find the indicated partial sum of each arithmetic series.

4. $\sum_{n=1}^{15}(3n - 1)$

5. $\sum_{k=1}^{20}(2k + 1)$

6. $\sum_{j=5}^{10}3j$

7. Identify the index, upper and lower limits, and general term of Item 4.

8. **Attend to precision.** Express the following sum using sigma notation: $3 + 7 + 11 + 15 + 19 + 23 + 27$

ACTIVITY 19 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 19-1

1. Determine whether or not each sequence is arithmetic. If the sequence is arithmetic, state the common difference.
a. 4, 5, 7, 10, . . .
b. 5, 7, 9, 11, . . .
c. 12, 9, 6, 3, . . .

2. Determine whether or not each sequence is arithmetic. If the sequence is arithmetic, use the explicit formula to write a general expression for a_n in terms of n.
a. 4, 12, 20, 28, . . .
b. 5, 10, 20, 40, . . .
c. 4, 0, −4, −8, . . .

3. Determine whether or not each sequence is arithmetic. If the sequence is arithmetic, use the recursive formula to write a general expression for a_n in terms of a_{n-1}.
a. 7, 7.5, 8, 8.5, . . .
b. 6, 7, 8, 9, . . .
c. −2, 4, −8, . . .

4. Find the indicated term of each arithmetic sequence.
a. $a_1 = 4$, $d = 5$; a_{15}
b. 14, 18, 22, 26, . . .; a_{20}
c. 45, 41, 37, 33, . . .; a_{18}

5. Find the sequence for which a_8 does NOT equal 24.
A. 3, 6, 9, . . .
B. −32, −24, −16, . . .
C. 108, 96, 84, . . .
D. −8, −4, 0, . . .

6. A radio station offers a $100 prize on the first day of a contest. Each day that the prize money is not awarded, $50 is added to the prize amount. If a contestant wins on the 17th day of the contest, how much money will be awarded?

7. If $a_4 = 20$ and $a_{12} = 68$, find a_1, a_2, and a_3.

8. Find the indicated term of each arithmetic sequence.
a. $a_1 = -2$, $d = 4$; a_{12}
b. 15, 19, 23, 27, . . .; a_{10}
c. 46, 40, 34, 28, . . .; a_{20}

9. What is the first value of n that corresponds to a positive value? Explain how you found your answer.

n	1	2	3	4	5
a_n	−42.5	−37.8	−33.1	−28.4	−23.7

10. Find the first four terms of the sequence with $a_1 = \frac{2}{3}$ and $a_n = a_{n-1} + \frac{1}{6}$.

11. If $a_1 = 3.1$ and $a_5 = -33.7$, write an expression for the sequence and find a_2, a_3, and a_4.

Lesson 19-2

12. Find the indicated partial sum of each arithmetic series.
a. $a_1 = 4$, $d = 5$; S_{10}
b. 14 + 18 + 22 + 26 + . . .; S_{12}
c. 45 + 41 + 37 + 33 + . . .; S_{18}

13. Find the indicated partial sum of each arithmetic series.
a. 1 + 3 + 5 + . . .; S_6
b. 1 + 3 + 5 + . . .; S_{10}
c. 1 + 3 + 5 + . . .; S_{12}
d. Explain the relationship between n and S_n in parts a–c.

14. Find the indicated partial sum of the arithmetic series.

$0 + (x + 2) + (2x + 4) + (3x + 6) + \ldots; S_{10}$

A. $9x + 18$
B. $10x + 20$
C. $45x + 90$
D. $55x + 110$

15. Two companies offer you a job. Company A offers you a $40,000 first-year salary with an annual raise of $1500. Company B offers you a $38,500 first-year salary with an annual raise of $2000.
a. What would your salary be with Company A as you begin your sixth year?
b. What would your salary be with Company B as you begin your sixth year?
c. What would be your total earnings with Company A after 5 years?
d. What would be your total earnings with Company B after 5 years?

16. If $S_{12} = 744$ and $a_1 = 40$, find d.

17. In an arithmetic series, $a_1 = 47$ and $a_7 = -13$, find d and S_7.

18. In an arithmetic series, $a_9 = 9.44$ and $d = 0.4$, find a_1 and S_9.

19. The first prize in a contest is \$500, the second prize is \$450, the third prize is \$400, and so on.
- **a.** How many prizes will be awarded if the last prize is \$100?
- **b.** How much money will be given out as prize money?

20. Find the sum of $13 + 25 + 37 + \ldots + 193$.
- **A.** 1339
- **B.** 1648
- **C.** 1930
- **D.** 2060

21. Find the sum of the first 150 natural numbers.

22. A store puts boxes of canned goods into a stacked display. There are 20 boxes in the bottom layer. Each layer has two fewer boxes than the layer below it. There are five layers of boxes. How many boxes are in the display? Explain your answer.

Lesson 19-3

23. Find the indicated partial sum of each arithmetic series.
- **a.** $\sum_{j=1}^{5}(5-6j)$
- **b.** $\sum_{j=1}^{20}5j$
- **c.** $\sum_{j=5}^{15}(5-j)$

24. Does $\sum_{j=1}^{10}(2j+1) = \sum_{j=1}^{5}(2j+1) + \sum_{j=6}^{10}(2j+1)$? Verify your answer.

25. Does $\sum_{j=4}^{9}(j-7) = \sum_{j=1}^{9}(j-7) - \sum_{j=1}^{3}(j-7)$? Verify your answer.

26. Which statement is true for the partial sum $\sum_{j=1}^{n}(4j+3)$?
- **A.** For $n = 5$, the sum is 35.
- **B.** For $n = 7$, the sum is 133.
- **C.** For $n = 10$, the sum is 230.
- **D.** For $n = 12$, the sum is 408.

27. Evaluate.
- **a.** $\sum_{j=1}^{6}(j+3)$
- **b.** $\sum_{j=10}^{15}(j-12)$
- **c.** $\sum_{j=1}^{8}(4j)$

28. Which is greater: $\sum_{j=4}^{8}(-3j+29)$ or $\sum_{j=4}^{8}-3j+29$?

29. Which expression is the sum of the series $7 + 10 + 13 + \ldots + 25$?
- **A.** $\sum_{j=1}^{7}4+3j$
- **B.** $\sum_{j=1}^{7}(4-3j)$
- **C.** $\sum_{j=1}^{7}(3+4j)$
- **D.** $\sum_{j=1}^{7}(4+3j)$

30. Evaluate $\sum_{j=1}^{5}\left(\frac{j \bullet \pi}{2}\right)$.

MATHEMATICAL PRACTICES
Look For and Make Use of Structure

31. How does the common difference of an arithmetic sequence relate to finding the partial sum of an arithmetic series?

Geometric Sequences and Series

ACTIVITY 20

Squares with Patterns

Lesson 20-1 Geometric Sequences

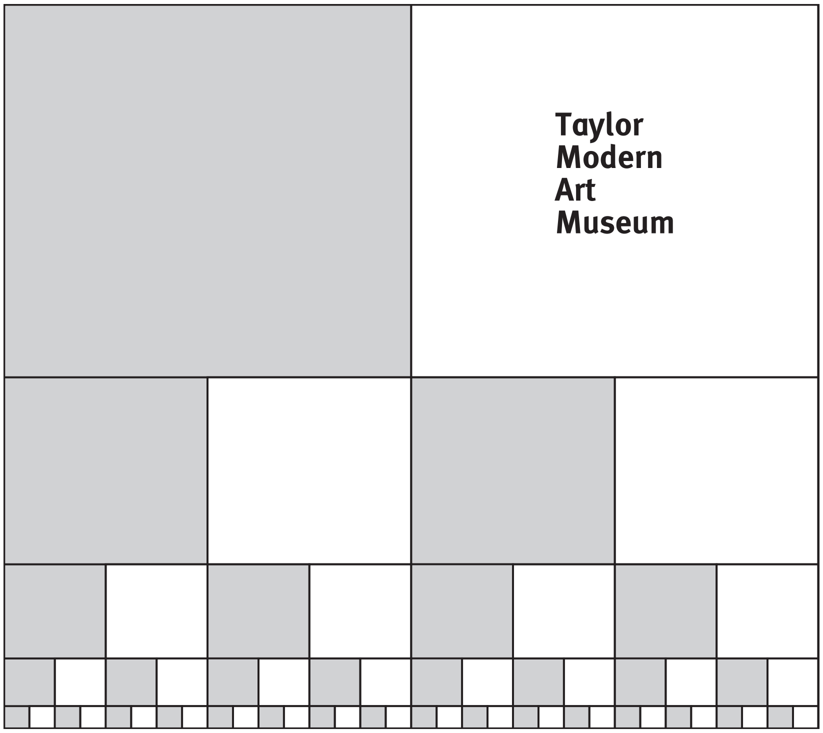
My Notes

Learning Targets:

- Determine whether a given sequence is geometric.
- Find the common ratio of a geometric sequence.
- Write an expression for a geometric sequence, and calculate the *n*th term.

SUGGESTED LEARNING STRATEGIES: Summarizing, Paraphrasing, Create Representations

Meredith is designing a mural for an outside wall of a warehouse that is being converted into the Taylor Modern Art Museum. The mural is 32 feet wide by 31 feet high. The design consists of squares in five different sizes that are painted black or white as shown below.



1. Let Square 1 be the largest size and Square 5 be the smallest size. For each size, record the length of the side, the number of squares of that size in the design, and the area of the square.

Square #	Side of Square (ft)	Number of Squares	Area of Square (ft^2)
1			
2			
3			
4			
5			

My Notes

2. Work with your group. Refer to the table in Item 1. As you share your ideas, be sure to use mathematical terms and academic vocabulary precisely. Make notes to help you remember the meaning of new words and how they are used to describe mathematical concepts.

a. Describe any patterns that you notice in the table.

b. Each column of numbers forms a sequence of numbers. List the four sequences that you see in the columns of the table.

c. Are any of those sequences arithmetic? Why or why not?

A ***geometric sequence*** is a sequence in which the ratio of consecutive terms is a constant. The constant is called the ***common ratio*** and is denoted by *r*.

MATH TIP

To find the common difference in an arithmetic sequence, subtract the preceding term from the following term.

To find the common ratio in a geometric sequence, divide any term by the preceding term.

3. Consider the sequences in Item 2b.

a. List those sequences that are geometric.

b. State the common ratio for each geometric sequence.

My Notes

4. Use a_n and a_{n-1} to write a general expression for the common ratio r.

5. Consider the sequences in the columns of the table in Item 1 that are labeled Square # and Side of Square.

a. Plot the Square # sequence by plotting the ordered pairs (term number, square number).

b. Using another color or symbol, plot the Side of Square sequence by plotting the ordered pairs (term number, side of square).

c. Is either sequence a linear function? Explain why or why not.

Check Your Understanding

6. Determine whether each sequence is arithmetic, geometric, or neither. If the sequence is arithmetic, state the common difference. If it is geometric, state the common ratio.

a. 3, 9, 27, 81, 243, . . .

b. 1, −2, 4, −8, 16, . . .

c. 4, 9, 16, 25, 36, . . .

d. 25, 20, 15, 10, 5, . . .

7. Use a_{n+1} and a_{n+2} to write an expression for the common ratio r.

8. Describe the graph of the first 5 terms of a geometric sequence with the first term 2 and the common ratio equal to 1.

9. Reason abstractly. Use the expression from Item 4 to write a *recursive formula* for the term a_n and describe what the formula means.

My Notes

The terms in a geometric sequence also can be written as the product of the first term and a power of the common ratio.

10. For the geometric sequence {4, 8, 16, 32, 64, … }, identify a_1 and r. Then fill in the missing exponents and blanks.

$a_1 =$ ________ $r =$ ________

$a_2 = 4 \cdot 2^{\_} = 8$

$a_3 = 4 \cdot 2^{\_} = 16$

$a_4 = 4 \cdot 2^{\_} =$ ________

$a_5 = 4 \cdot 2^{\_} =$ ________

$a_6 = 4 \cdot 2^{\_} =$ ________

$a_{10} = 4 \cdot 2^{\_} =$ ________

11. Use a_1, r, and n to write an *explicit formula* for the nth term of any geometric sequence.

12. Use the formula from Item 11 to find the indicated term in each geometric sequence.

a. 1, 2, 4, 8, 16, . . . ; a_{16}

b. 4096, 1024, 256, 64, . . . ; a_9

My Notes

Check Your Understanding

13. a. Complete the table for the terms in the sequence with $a_1 = 3$; $r = 2$.

Term	Recursive $a_n = a_{n-1} \cdot r$	Explicit $a_n = a_1 \cdot r^{n-1}$	Value of Term
a_1	3	$3 \cdot 2^{1-1} = 3$	3
a_2	$3 \cdot 2$	$3 \cdot 2^{2-1} = 3 \cdot 2$	6
a_3	$(3 \cdot 2) \cdot 2$	$3 \cdot 2^{3-1} = 3 \cdot 2^2$	12
a_4			
a_5			

b. What does the product $(3 \cdot 2)$ represent in the recursive expression for a_3?

c. Express regularity in repeated reasoning. Compare the recursive and explicit expressions for each term. What do you notice?

LESSON 20-1 PRACTICE

14. Write a formula that will produce the sequence that appears on the calculator screen below.

```
5*3
                15
Ans*3
                45
               135
               405
```

15. Determine whether each sequence is arithmetic, geometric, or neither. If the sequence is arithmetic, state the common difference, and if it is geometric, state the common ratio.

a. 3, 5, 7, 9, 11, . . .

b. 5, 15, 45, 135, . . .

c. $6, -4, \frac{8}{3}, -\frac{16}{9}, \ldots$

d. 1, 2, 4, 7, 11, . . .

16. Find the indicated term of each geometric sequence.

a. $a_1 = -2, r = 3; a_8$

b. $a_1 = 1024, r = -\frac{1}{2}; a_{12}$

17. Attend to precision. Given the data in the table below, write both a recursive formula and an explicit formula for a_n.

n	1	2	3	4
a_n	0.25	0.75	2.25	6.75

My Notes

MATH TERMS

A **finite series** is the sum of a finite sequence and has a specific number of terms.

An ***infinite series*** is the sum of an infinite sequence and has an infinite number of terms. You will work with infinite series later in this Lesson.

MATH TIP

When writing out a *sequence*, separate the terms with commas. A *series* is written out as an expression and the terms are separated by addition symbols. If a series has negative terms, then the series may be written with subtraction symbols.

Learning Targets:

- Derive the formula for the sum of a finite geometric series.
- Calculate the partial sums of a geometric series.

SUGGESTED LEARNING STRATEGIES: Close Reading, Vocabulary Organizer, Think-Pair-Share, Create Representations

The sum of the terms of a geometric sequence is a ***geometric series***. The sum of a ***finite geometric series*** where $r \neq 1$ is given by these formulas:

$$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n-1}$$

$$S_n = a_1\left(\frac{1-r^n}{1-r}\right)$$

1. To derive the formula, Step 1 requires multiplying the equation of the sum by $-r$. Follow the remaining steps on the left to complete the derivation of the sum formula.

Step 1 $S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n-1}$
$-rS_n = -a_1 r - a_1 r^2 - a_1 r^3 - \ldots - a_1 r^{n-1} - a_1 r^n$

Step 2 Combine terms on each side of the equation (most terms will cancel out).

Step 3 Factor out S_n on the left side of the equation and factor out a_1 on the right.

Step 4 Solve for S_n.

Example A

Find the total of the Area of Square column in the table in Item 1 from the last lesson. Then use the formula developed in Item 1 of this lesson to find the total area and show that the result is the same.

Step 1: Add the areas of each square from the table.

$256 + 64 + 16 + 4 + 1 = 341$

Square #	1	2	3	4	5
Area	256	64	16	4	1

Step 2: Find the common ratio.

$\frac{64}{256} = 0.25, \frac{16}{64} = 0.25, \frac{4}{16} = 0.25; r = 0.25$

Step 3: Substitute $n = 5$, $a_1 = 256$, and $r = 0.25$ into the formula for S_n.

$S_n = a_1\left(\frac{1-r^n}{1-r}\right); S_5 = 256\left(\frac{1-0.25^5}{1-0.25}\right)$

Step 4: Evaluate S_5.

$S_5 = 256\left(\frac{1-0.25^5}{1-0.25}\right) = 341$

My Notes

Try These A

Find the indicated sum of each geometric series. Show your work.

a. Find S_5 for the geometric series with $a_1 = 5$ and $r = 2$.

b. $256 + 64 + 16 + 4 + \ldots; S_6$

c. $\sum_{n=1}^{10} 2 \cdot 3^{n-1}$

Check Your Understanding

2. Reason quantitatively. How do you determine if the common ratio in a series is negative?

3. Find the sum of the series $2 + 8 + 32 + 128 + 512$ using sigma notation.

MATH TIP

Recall that *sigma notation* is a shorthand notation for a series. For example:

$$\sum_{n=1}^{3} 8 \cdot 2^{n-1}$$
$$= 8(2)^{(1-1)} + 8(2)^{(2-1)} + 8(2)^{(3-1)}$$
$$= 8 \cdot 1 + 8 \cdot 2 + 8 \cdot 4$$
$$= 8 + 16 + 32$$
$$= 56$$

Recall that the sum of the first n terms of a series is a *partial sum*. For some geometric series, the partial sums $S_1, S_2, S_3, S_4, \ldots$ form a sequence with terms that approach a limiting value. The limiting value is called the ***sum of the infinite geometric series***.

To understand the concept of an infinite sum of a geometric series, follow these steps.

- Start with a square piece of paper, and let it represent one whole unit.
- Cut the paper in half, place one piece of the paper on your desk, and keep the other piece of paper in your hand. The paper on your desk represents the first partial sum of the series, $S_1 = \frac{1}{2}$.
- Cut the paper in your hand in half again, adding one of the pieces to the paper on your desk and keeping the other piece in your hand. The paper on your desk now represents the second partial sum.
- Repeat this process as many times as you are able.

MATH TIP

If the terms in the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ get close to some constant as n gets very large, the constant is the limiting value of the sequence. For example, in the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{n}, \ldots,$ the terms get closer to a limiting value of 0 as n gets larger.

4. Use appropriate tools strategically. Each time you add a piece of paper to your desk, the paper represents the next term in the geometric series.

a. As you continue the process of placing half of the remaining paper on your desk, what happens to the amount of paper on your desktop?

My Notes

CONNECT TO AP

An infinite series whose partial sums continually get closer to a specific number is said to *converge*, and that number is called the *sum of the infinite series.*

4. b. Fill in the blanks to complete the partial sums for the infinite geometric series represented by the pieces of paper on your desk.

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \_\_\_ = \_\_\_$

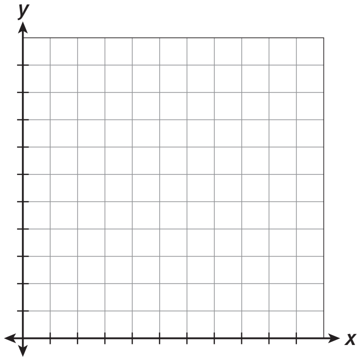
$S_3 = \frac{1}{2} + \_\_\_ + \_\_\_ = \_\_\_$

$S_4 = \frac{1}{2} + \_\_\_ + \_\_\_ + \_\_\_ = \_\_\_$

$S_5 = \frac{1}{2} + \_\_\_ + \_\_\_ + \_\_\_ + \_\_\_ = \_\_\_$

$S_6 = \frac{1}{2} + \_\_\_ + \_\_\_ + \_\_\_ + \_\_\_ + \_\_\_ = \_\_\_$

c. Plot the first six partial sums.



d. Do the partial sums appear to be approaching a limiting value? If so, what does the value appear to be?

My Notes

5. Consider the geometric series $2 + 4 + 8 + 16 + 32 + \ldots.$

a. List the first five partial sums for this series.

b. Do these partial sums appear to have a limiting value?

c. Does there appear to be a sum of the infinite series? If so, what does the sum appear to be? If not, why not?

6. Consider the geometric series $3 - 1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \frac{1}{243} - \ldots$

a. List the first seven partial sums for this series.

b. Do these partial sums appear to have a limiting value?

c. Does there appear to be a sum of the infinite series? If so, what does the sum appear to be? If not, why not?

WRITING MATH

You can write the sum of an infinite series by using summation, or sigma, notation and using an infinity symbol for the upper limit. For example,

$$\sum_{n=1}^{\infty} 3\left(-\frac{1}{3}\right)^{n-1}$$

$$= 3 - 1 + \frac{1}{3} - \ldots$$

My Notes

Check Your Understanding

Find the indicated partial sums of each geometric series. Do these partial sums appear to have a limiting value? If so, what does the infinite sum appear to be?

7. First 8 partial sums of the series $1 + 2 + 4 + 8 + \ldots$

8. First 6 partial sums of the series $\frac{2}{5} + \frac{2}{15} + \frac{2}{45} + \frac{2}{135} + \ldots$

LESSON 20-2 PRACTICE

Find the indicated partial sum of each geometric series.

9. $1 - 3 + 9 - 27 + \ldots; S_7$

10. $\frac{1}{625} - \frac{1}{125} + \frac{1}{25} - \frac{1}{5} + \ldots; S_9$

Consider the geometric series $-1 + 1 - 1 + 1 - 1 + \ldots$

11. Find S_4 and S_6. Generalize the partial sum when n is an even number.

12. Find S_5 and S_7. Generalize the partial sum when n is on odd number.

13. Describe any conclusions drawn from Items 11 and 12.

14. **Construct viable arguments.** What conclusions if any can you draw from this lesson about the partial sums of geometric series where $r \geq 1$ or $r \leq -1$?

My Notes

Learning Targets:

- Determine if an infinite geometric sum converges.
- Find the sum of a convergent geometric series.

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Quickwrite

Recall the formula for the sum of a finite series $S_n = \frac{a_1(1-r^n)}{1-r}$. To find the sum of an infinite series, find the value that S_n gets close to as n gets very large. For any infinite geometric series where $-1 < r < 1$, as n gets very large, r^n gets close to 0.

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S \approx \frac{a_1(1-0)}{1-r}$$

$$\approx \frac{a_1}{1-r}$$

An infinite geometric series $\sum_{n=0}^{\infty} a_1 r^n$ converges to the sum $S = \frac{a_1}{1-r}$ if and only if $|r| < 1$ or $-1 < r < 1$. If $|r| \geq 1$, the infinite sum does not exist.

MATH TIP

S_n represents the sum of a finite series. Use S to indicate the sum of an infinite series.

MATH TIP

$-1 < r < 1$ can be written as $|r| < 1$. As n increases, r^n gets close to, or *approaches*, 0. It is important to realize that as r^n approaches 0, you can say that $|r^n|$, but not r^n, is getting "smaller."

1. Consider the three series from Items 4–6 of the previous lesson. Decide whether the formula for the sum of an infinite geometric series applies. If so, use it to find the sum. Compare the results to your previous answers.

a. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$

b. $2 + 4 + 8 + 16 + 32 + \ldots$

c. $3 - 1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \frac{1}{243} - \ldots$

Check Your Understanding

Find the infinite sum if it exists or tell why it does not exist. Show your work.

2. $64 + 16 + 4 + 1 + \ldots$

3. $\frac{1}{3} + \frac{5}{12} + \frac{25}{48} + \frac{125}{192} + \ldots$

4. $\sum_{n=1}^{\infty} 3\left(\frac{2}{5}\right)^{n-1}$

My Notes

5. Consider the arithmetic series $2 + 5 + 8 + 11 + \ldots$.

a. Find the first four partial sums of the series.

b. Do these partial sums appear to have a limiting value?

c. Does the arithmetic series appear to have an infinite sum? Explain.

6. Summarize the following formulas for a geometric series.

common ratio	$r =$ ____________
*n*th term	$a_n =$ ____________
Sum of first *n* terms	$S_n =$ ____________
Infinite sum	$S =$ ____________

Check Your Understanding

Consider the series $0.2 + 0.02 + 0.002 + \ldots$.

7. Find the common ratio between the terms of the series.

8. Does this series have an infinite sum? If yes, use the formula to find the sum.

9. **Construct viable arguments.** Make a conjecture about the infinite sum $0.5 + 0.05 + 0.005 + \ldots$. Then verify your conjecture with the formula.

LESSON 20-3 PRACTICE

Find the infinite sum if it exists, or tell why it does not exist.

10. $18 - 9 + \frac{9}{2} - \frac{9}{4} + \ldots$

11. $729 + 486 + 324 + 216 + \ldots$

12. $81 + 108 + 144 + 192 + \ldots$

13. $-33 - 66 - 99 - 132 - \ldots$

14. **Reason quantitatively.** At the beginning of the lesson it is stated that "for any infinite geometric series where $-1 < r < 1$, as n gets very large, r^n gets close to 0." Justify this statement with an example, using a negative value for r.

ACTIVITY 20 PRACTICE

Write your answers on notebook paper.
Show your work.

Lesson 20-1

1. Write *arithmetic, geometric,* or *neither* for each sequence. If arithmetic, state the common difference. If geometric, state the common ratio.
   a. 4, 12, 36, 108, 324, . . .
   b. 1, 2, 6, 24, 120, . . .
   c. 4, 9, 14, 19, 24, . . .
   d. 35, −30, 25, −20, 15, . . .

2. Find the indicated term of each geometric series.
   a. $a_1 = 1, r = -3; a_{10}$
   b. $a_1 = 3072, r = \frac{1}{4}; a_8$

3. If a_n is a geometric sequence, express the quotient of $\frac{a_7}{a_4}$ in terms of r.

4. The first three terms of a geometric series are $\frac{1}{81}, \frac{1}{27}, \frac{1}{9}, \ldots$ What is a_6?
   A. $\frac{3}{81}$
   B. 3
   C. $\frac{364}{81}$
   D. 9

5. Determine the first three terms of a geometric sequence with a common ratio of 2 and defined as follows:
   $x - 1, x + 6, 3x + 4$

6. Determine whether each sequence is geometric. If it is a geometric sequence, state the common ratio.
   a. $x, x^2, x^4, \ldots$
   b. $(x + 3), (x + 3)^2, (x + 3)^3, \ldots$
   c. $3^x, 3^{x+1}, 3^{x+2}, \ldots$
   d. $x^2, (2x)^2, (3x)^2, \ldots$

7. If $a_3 = \frac{9}{32}$ and $a_5 = \frac{81}{512}$, find a_1 and r.

8. The 5 in the expression $a_n = 4(5)^{n-1}$ represents which part of the expression?
   A. n
   B. a_1
   C. r
   D. S_n

9. A ball is dropped from a height of 24 feet. The ball bounces to 85% of its previous height with each bounce. Write an expression and solve to find how high (to the nearest tenth of a foot) the ball bounces on the sixth bounce.

10. Write the recursive formula for each sequence.
   a. 4, 2, 1, 0.5, . . .
   b. 2, 6, 18, 54, 162, . . .
   c. $\frac{4}{5}, \frac{4}{25}, \frac{4}{125}, \ldots$
   d. $-45, 5, -\frac{5}{9}, \ldots$

11. Write the explicit formula for each sequence.
   a. 4, 2, 1, 0.5, . . .
   b. 2, 6, 18, 54, 162, . . .
   c. $\frac{4}{5}, \frac{4}{25}, \frac{4}{125}, \ldots$
   d. $-45, 5, -\frac{5}{9}, \ldots$

Lesson 20-2

12. Find the indicated partial sum of each geometric series.
   a. $5 + 2 + \frac{4}{5} + \frac{8}{25} + \ldots; S_7$
   b. $\frac{1}{8} + \frac{1}{4} + \frac{1}{2} + \ldots; S_{15}$

13. For the geometric series 2.9 + 3.77 + 4.90 + 6.37 + . . . , do the following:
   a. Find S_9 (to the nearest hundredth).
   b. How many more terms have to be added in order for the sum to be greater than 200?

14. George and Martha had two children by 1776, and each child had two children. If this pattern continued to the 12th generation, how many descendants do George and Martha have?

15. A finite geometric series is defined as 0.6 + 0.84 + 1.18 + 1.65 + . . . + 17.36. How many terms are in the series?
   A. $n = 5$
   B. $n = 8$
   C. $n = 10$
   D. $n = 11$

16. Evaluate $\sum_{j=1}^{6} 3(2)^j$

17. During a 10-week summer promotion, a baseball team is letting all spectators enter their names in a weekly drawing each time they purchase a game ticket. Once a name is in the drawing, it remains in the drawing unless it is chosen as a winner. Since the number of names in the drawing increases each week, so does the prize money. The first week of the contest the prize amount is \$10, and it doubles each week.
  a. What is the prize amount in the fourth week of the contest? In the tenth week?
  b. What is the total amount of money given away during the entire promotion?

18. In case of a school closing due to inclement weather, the high school staff has a calling system to make certain that everyone is notified. In the first round of phone calls, the principal calls three staff members. In the second round of calls, each of those three staff members calls three more staff members. The process continues until all of the staff is notified.
  a. Write a rule that shows how many staff members are called during the *n*th round of calls.
  b. Find the number of staff members called during the fourth round of calls.
  c. If all of the staff has been notified after the fourth round of calls, how many people are on staff at the high school, including the principal?

Lesson 20-3

19. Find the infinite sum if it exists. If it does not exist, tell why.
  a. $24 + 12 + 6 + 3 + \ldots$
  b. $\frac{1}{12} + \frac{1}{6} + \frac{1}{3} + \frac{2}{3} + \ldots$
  c. $1296 - 216 + 36 - 6 + \ldots$

20. Write an expression in terms of a_n that means the same as $\sum_{j=1}^{\infty} 2\left(\frac{1}{3}\right)^j$

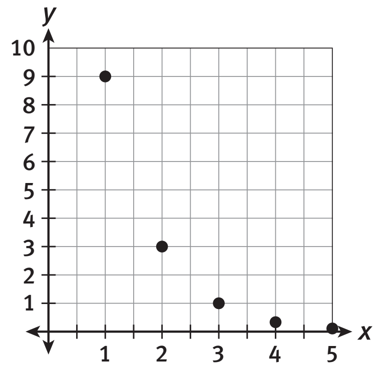
21. Express 0.2727 . . . as a fraction.

22. Use the common ratio to determine if the infinite series converges or diverges.
  a. $36 + 24 + 12 + \ldots$
  b. $-4 + 2 + (-1) + \ldots$
  c. $3 + 4.5 + 6.75 + \ldots$

23. The infinite sum $0.1 + 0.05 + 0.025 + 0.0125 + \ldots$
  A. diverges.
  B. converges at 0.2.
  C. converges at 0.5.
  D. converges at 1.0.

24. An infinite geometric series has $a_1 = 3$ and a sum of 4. Find r.

25. The graph depicts which of the following?



  A. converging arithmetic series
  B. converging geometric series
  C. diverging arithmetic series
  D. diverging geometric series

26. True or false? No arithmetic series with a common difference that is not equal to zero has an infinite sum. Explain.

MATHEMATICAL PRACTICES
Make Sense of Problems and Persevere in Solving Them

27. Explain how knowing any two terms of a geometric sequence is sufficient for finding the other terms.

Sequences and Series
THE CHESSBOARD PROBLEM

In a classic math problem, a king wants to reward a knight who has rescued him from an attack. The king gives the knight a chessboard and plans to place money on each square. He gives the knight two options. Option 1 is to place a thousand dollars on the first square, two thousand on the second square, three thousand on the third square, and so on. Option 2 is to place one penny on the first square, two pennies on the second, four on the third, and so on.

Think about which offer sounds better and then answer these questions.

1. List the first five terms in the sequences formed by the given options. Identify each sequence as arithmetic, geometric, or neither.
   a. Option 1
   b. Option 2

2. For each option, write a rule that tells how much money is placed on the nth square of the chessboard and a rule that tells the total amount of money placed on squares 1 through n.
   a. Option 1
   b. Option 2

3. Find the amount of money placed on the 20th square of the chessboard and the total amount placed on squares 1 through 20 for each option.
   a. Option 1
   b. Option 2

4. There are 64 squares on a chessboard. Find the total amount of money placed on the chessboard for each option.
   a. Option 1
   b. Option 2

5. Which gives the better reward, Option 1 or Option 2? Explain why.

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
	The solution demonstrates these characteristics:			
Mathematics Knowledge and Thinking (Items 1, 3, 4)	• Fluency in determining specified terms of a sequence or the sum of a specific number of terms of a series	• A functional understanding and accurate identification of specified terms of a sequence or the sum of a specific number of terms of a series	• Partial understanding and partially accurate identification of specified terms of a sequence or the sum of a specific number of terms of a series	• Little or no understanding and inaccurate identification of specified terms of a sequence or the sum of a specific number of terms of a series
Problem Solving (Items 3, 4)	• An appropriate and efficient strategy that results in a correct answer	• A strategy that may include unnecessary steps but results in a correct answer	• A strategy that results in some incorrect answers	• No clear strategy when solving problems
Mathematical Modeling / Representations (Items 1, 2)	• Fluency in accurately representing real-world scenarios with arithmetic and geometric sequences and series	• Little difficulty in accurately representing real-world scenarios with arithmetic and geometric sequences and series	• Some difficulty in representing real-world scenarios with arithmetic and geometric sequences and series	• Significant difficulty in representing real-world scenarios with arithmetic and geometric sequences and series
Reasoning and Communication (Item 5)	• Clear and accurate explanation of which option provides the better reward	• Adequate explanation of which option provides the better reward	• Misleading or confusing explanation of which option provides the better reward	• Incomplete or inadequate explanation of which option provides the better reward

Exponential Functions and Graphs

ACTIVITY 21

Sizing Up the Situation

Lesson 21-1 Exploring Exponential Patterns

My Notes

Learning Targets:

- Identify data that grow exponentially.
- Compare the rates of change of linear and exponential data.

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Quickwrite

Ramon Hall, a graphic artist, needs to make several different-sized draft copies of an original design. His original graphic design sketch is contained within a rectangle with a width of 4 cm and a length of 6 cm. Using the office copy machine, he magnifies the original 4 cm × 6 cm design to 120% of the original design size, and calls this his first draft. Ramon's second draft results from magnifying the first draft to 120% of its new size. Each new draft is 120% of the previous draft.

1. Complete the table with the dimensions of Ramon's first five draft versions, showing all decimal places.

Number of Magnifications	Width (cm)	Length (cm)
0	4	6
1		
2		
3		
4		
5		

MATH TIP

Magnifying a design creates similar figures. The ratio between corresponding lengths of similar figures is called the *constant of proportionality*, or the *scale factor*. For a magnification of 120%, the scale factor is 1.2.

2. **Make sense of problems.** The resulting draft for each magnification has a unique width and a unique length. Thus, there is a functional relationship between the number of magnifications n and the resulting width W. There is also a functional relationship between the number of magnifications n and the resulting length L. What are the reasonable domain and range for these functions? Explain.

3. Plot the ordered pairs (n, W) from the table in Item 1. Use a different color or symbol to plot the ordered pairs (n, L).

My Notes

4. Use the data in Item 1 to complete the table.

Increase in Number of Magnifications	Change in the Width	Change in the Length
0 to 1	$4.8 - 4 = 0.8$	$7.2 - 6 = 1.2$
1 to 2		
2 to 3		
3 to 4		
4 to 5		

5. From the graphs in Item 3 and the data in Item 4, do these functions appear to be linear? Explain why or why not.

MATH TIP

Linear functions have the property that the rate of change of the output variable *y* with respect to the input variable *x* is constant, that is, the ratio $\frac{\Delta y}{\Delta x}$ is constant for linear functions.

6. **Express regularity in repeated reasoning.** Explain why each table below contains data that can be represented by a linear function. Write an equation to show the linear relationship between *x* and *y*.

a.

x	−3	−1	1	3	5
y	8	5	2	−1	−4

b.

x	2	5	11	17	26
y	3	7	15	23	35

7. Consider the data in the table below.

x	0	1	2	3	4
y	24	12	6	3	1.5

a. Can the data in the table be represented by a linear function? Explain why or why not.

b. Describe any patterns that you see in the consecutive *y*-values.

My Notes

8. Consider the data in the table in Item 1. How does the relationship of the data in this table compare to the relationship of the data in the table in Item 7?

Check Your Understanding

9. Complete the table so that the function represented is a linear function.

x	1	2	3	4	5
$f(x)$	16	22			40

10. **Reason quantitatively.** Explain why the function represented in the table cannot be a linear function.

x	1	2	3	4	5
$f(x)$	7	12	16	19	21

LESSON 21-1 PRACTICE

Model with mathematics. Determine whether each function is linear or nonlinear. Explain your answers.

11. x = number of equally sized pans of brownies; $f(x)$ = number of brownies

12. x = cost of an item; $f(x)$ = price you pay in a state with a 6% sales tax

13. x = number of months; $f(x)$ = amount of money in a bank account with interest compounded monthly

14.

x	2	4	6	8	10
y	2.6	3.0	3.8	4.8	6.0

15.

x	5	10	15	20	25
y	1.25	1.00	0.75	0.50	0.25

16. Identify if there is a constant rate of change or constant multiplier. Determine the rate of change or constant multiplier.

x	1	2	3	4
y	6	4.8	3.84	3.072

My Notes

Learning Targets:

- Identify and write exponential functions.
- Determine the decay factor or growth factor of an exponential function.

SUGGESTED LEARNING STRATEGIES: Vocabulary Organizer, Create Representations, Look for a Pattern, Quickwrite, Think-Pair-Share

The data in the tables in Items 7 and 8 of the previous lesson were generated by ***exponential functions***. In the special case when the change in the input variable x is constant, the output variable y of an exponential function changes by a *multiplicative constant*. For example, in the table in Item 7, the increase in the consecutive x-values results from repeatedly adding 1, while the decrease in y-values results from repeatedly multiplying by the constant $\frac{1}{2}$, known as the ***exponential decay factor***.

MATH TERMS

An **exponential function** is a function of the form $f(x) = a \cdot b^x$, where a and b are constants, x is the domain, $f(x)$ is the range, $a \neq 0$, $b \neq 0$, and $b \neq 1$.

1. In the table in Item 1 in Lesson 21-1, what is the ***exponential growth factor***?

MATH TERMS

In an exponential function, the multiplicative constant is called an **exponential decay factor** when it is between 0 and 1.

When the multiplicative constant is greater than 1, it is called an **exponential growth factor**.

2. You can write an equation for the exponential function relating W and n.

a. Complete the table below to show the calculations used to find the width of each magnification.

Number of Magnifications	Calculation to Find Width (cm)
0	4
1	4(1.2)
2	4(1.2)(1.2)
3	4(1.2)(1.2)(1.2)
4	
5	
10	
n	

MATH TIP

To compare change in size, you could also use the *growth rate*, or *percent increase*. This is the percent that is equal to the ratio of the increase amount to the original amount.

b. Express regularity in repeated reasoning. Write a function that expresses the resulting width W after n magnifications of 120%.

CONNECT TO TECHNOLOGY

Confirm the reasonableness of your function in Item 2b by using a graphing calculator to make a scatter plot of the data in the table in Item 8 in Lesson 21-1. Then graph the function to see how it compares to the scatter plot.

c. Use the function in part b to find the width of the 11th magnification.

My Notes

The general form of an exponential function is $f(x) = a(b^x)$, where a and b are constants and $a \neq 0$, $b > 0$, $b \neq 1$.

3. For the exponential function written in Item 2b, identify the value of the parameters a and b. Then explain their meaning in terms of the problem situation.

4. Starting with Ramon's original 4 cm × 6 cm rectangle containing his graphic design, write an exponential function that expresses the resulting length L after n magnifications of 120%.

Ramon decides to print five different reduced draft copies of his original design rectangle. Each one will be reduced to 90% of the previous size.

5. Complete the table below to show the dimensions of the first five draft versions. Include all decimal places.

Number of Reductions	Width (cm)	Length (cm)
0	4	6
1		
2		
3		
4		
5		

6. Write the exponential decay factor and the *decay rate* for the data in the table in Item 5.

MATH TIP

To compare change in size, you could also use the *decay rate*, or *percent decrease*. This is the percent that is equal to the ratio of the decrease amount to the original amount.

7. Model with mathematics. Use the data in the table in Item 5.

a. Write an exponential function that expresses the width w of a reduction in terms of n, the number of reductions performed.

b. Write an exponential function that expresses the length l of a reduction in terms of n, the number of reductions performed.

c. Use the functions to find the dimensions of the design if the original design undergoes ten reductions.

My Notes

Check Your Understanding

8. Why is it necessary to place restrictions that $a \neq 0$, $b > 0$, and $b \neq 1$ in the general form of an exponential function?

9. An exponential function contains the ordered pairs (3, 6), (4, 12), and (5, 24).

a. What is the scale factor for this function?

b. Does the function represent exponential decay or growth? Explain your reasoning.

10. **Make sense of problems.** For the equation $y = 2000(1.05)^x$, identify the value of the parameters a and b. Then explain their meaning in terms of a savings account in a bank.

LESSON 21-2 PRACTICE

Construct viable arguments. Decide whether each table of data can be modeled by a linear function, an exponential function, or neither, and justify your answers. If the data can be modeled by a linear or exponential function, give an equation for the function using regression methods available through technology.

11.

x	0	1	2	3	4
y	1	3	9	27	81

12.

x	0	1	2	3	4
y	4	8	14	22	32

13. Given that the function has an exponential decay factor of 0.8, complete the table.

x	0	1	2	3	4
y	64				

14. What is the decay rate for the function in Item 13?

15. Write the function represented in Item 13.

My Notes

Learning Targets:

- Determine when an exponential function is increasing or decreasing.
- Describe the end behavior of exponential functions.
- Identify asymptotes of exponential functions.

SUGGESTED LEARNING STRATEGIES: Create Representations, Activating Prior Knowledge, Close Reading, Vocabulary Organizer, Think-Pair-Share, Group Presentation

1. Graph the functions $y = 6(1.2)^x$ and $y = 6(0.9)^x$ on a graphing calculator or other graphing utility. Sketch the results.

2. Determine the domain and range for each function. Use interval notation.

	Domain	Range
a. $y = 6(1.2)^x$		
b. $y = 6(0.9)^x$		

A function is said to *increase* if the *y*-values increase as the *x*-values increase. A function is said to *decrease* if the *y*-values decrease as the *x*-values increase.

3. Describe each function as increasing or decreasing.

a. $y = 6(1.2)^x$

b. $y = 6(0.9)^x$

CONNECT TO AP

Not all functions increase or decrease over the entire domain of the function. Functions may increase, decrease, or remain constant over various intervals of the domain. Functions that either increase or decrease over the entire domain are called *strictly monotonic*.

As you learned in a previous activity, the *end behavior* of a graph describes the *y*-values of the function as *x* increases without bound and as *x* decreases without bound. If the end behavior approaches some constant *a*, then the graph of the function has a horizontal ***asymptote*** at $y = a$.

When *x* increases without bound, the values of *x* approach positive infinity, ∞. When *x* decreases without bound, the values of *x* approach negative infinity, $-\infty$.

MATH TERMS

If the graph of a relation gets closer and closer to a line, the line is called an **asymptote** of the graph.

4. Describe the end behavior of each function as *x* approaches ∞. Write the equation for any horizontal asymptotes.

a. $y = 6(1.2)^x$

b. $y = 6(0.9)^x$

My Notes

5. Describe the end behavior of each function as x approaches $-\infty$. Write the equation for any horizontal asymptotes.
   a. $y = 6(1.2)^x$

   b. $y = 6(0.9)^x$

6. Identify any x- or y-intercepts of each function.
   a. $y = 6(1.2)^x$

   b. $y = 6(0.9)^x$

7. **Reason abstractly.** Consider how the parameters a and b affect the graph of the general exponential function $f(x) = a(b)^x$. In parts a–c, use a graphing calculator to graph each of the following functions. Compare and contrast the graphs.
   a. $f(x) = 2^x$; $g(x) = 3(2)^x$; $h(x) = -3(2)^x$; $j(x) = \frac{1}{4}(2)^x$; $k(x) = -\frac{1}{4}(2)^x$

   b. $f(x) = 10^x$; $g(x) = 2(10)^x$; $h(x) = -3(10)^x$; $j(x) = \frac{1}{4}(10)^x$; $k(x) = -\frac{1}{4}(10)^x$

   c. $f(x) = \left(\frac{1}{2}\right)^x$; $g(x) = 4\left(\frac{1}{2}\right)^x$; $h(x) = -6\left(\frac{1}{4}\right)^x$; $j(x) = \frac{1}{2}\left(\frac{1}{4}\right)^x$;
   $k(x) = -\frac{1}{4}\left(\frac{1}{10}\right)^x$

   d. Describe the effects of different values of a and b in the general exponential function $f(x) = a(b)^x$. Consider attributes of the graph such as the y-intercept, horizontal asymptotes, and whether the graph is increasing or decreasing.

Check Your Understanding

Graph the functions $f(x) = -6(1.2)^x$ and $g(x) = -6(0.9)^x$ on a graphing calculator or other graphing utility.

8. Determine the domain and range for each function.
9. Describe the end behavior of each function as x approaches ∞.
10. Describe the end behavior of each function as x approaches $-\infty$.

LESSON 21-3 PRACTICE

Make use of structure. For each exponential function, state whether the function increases or decreases, and give the y-intercept. Use the general form of an exponential function to explain your answers.

11. $y = 8(2)^x$
12. $y = 0.3(0.25)^x$
13. $y = -2(10)^x$
14. $y = -(0.3)^x$
15. **Construct viable arguments.** What is true about the asymptotes and y-intercepts of the functions in this lesson? What conclusions can you draw?

My Notes

Learning Targets:

- Explore how changing parameters affects the graph of an exponential function.
- Graph transformations of exponential functions.

SUGGESTED LEARNING STRATEGIES: Close Reading, Create Representations, Quickwrite

You can use transformations of the graph of the function $f(x) = b^x$ to graph functions of the form $g(x) = a(b)^{x-c} + d$, where a and b are constants, and $a \neq 0$, $b > 0$, $b \neq 1$. Rather than having a single parent graph for all exponential functions, there is a different parent graph for each base b.

MATH TIP

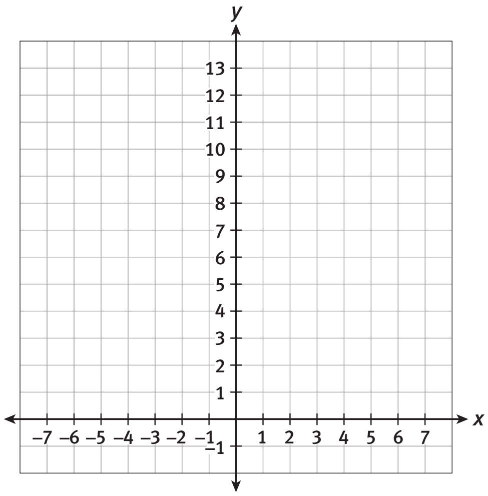
You can draw a quick sketch of the parent graph for any base b by plotting the points $\left(-1, \frac{1}{b}\right)$, (0, 1), and (1, b).

CONNECT TO AP

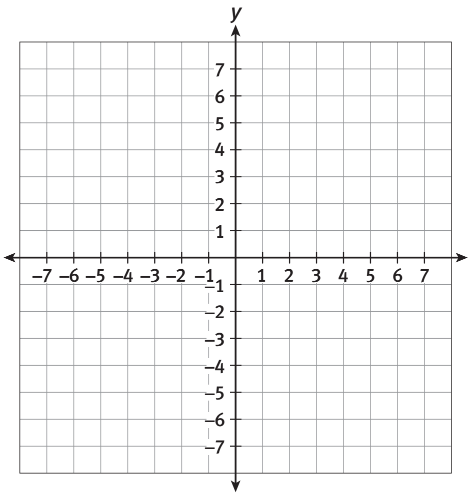
Exponential functions are important in the study of calculus.

1. Graph the parent graph f and the function g by applying the correct vertical stretch, shrink, and/or reflection over the x-axis. Write a description of each transformation.

a. $f(x) = \left(\frac{1}{2}\right)^x$ $\qquad g(x) = 4\left(\frac{1}{2}\right)^x$



b. $f(x) = 3^x$ $\qquad g(x) = -\frac{1}{2}(3)^x$



My Notes

CONNECT TO TECHNOLOGY

You can use a graphing calculator to approximate the range values when the *x*-coordinates are not integers. For $f(x) = 2^x$, use a calculator to find $f\left(\frac{1}{2}\right)$ and $f\left(\sqrt{3}\right)$.

$$2^{\frac{1}{2}} \approx 1.414$$
$$2^{\sqrt{3}} \approx 3.322$$

Then use a graphing calculator to verify that the points $\left(\frac{1}{2}, 2^{\frac{1}{2}}\right)$ and $\left(\sqrt{3}, 2^{\sqrt{3}}\right)$ lie on the graph of $f(x) = 2^x$.

2. Sketch the parent graph *f* and the graphs of *g* and *h* by applying the correct horizontal or vertical translation. Write a description of each transformation and give the equations of any asymptotes.

a. $f(x) = 2^x$
$g(x) = 2^{(x-3)}$
$h(x) = 2^{(x+2)}$

b. $f(x) = 10^x$
$g(x) = 10^{(x-1)}$
$h(x) = 10^{(x+3)}$

c. $f(x) = 10^x$
$g(x) = 10^x - 1$
$h(x) = 10^x + 3$

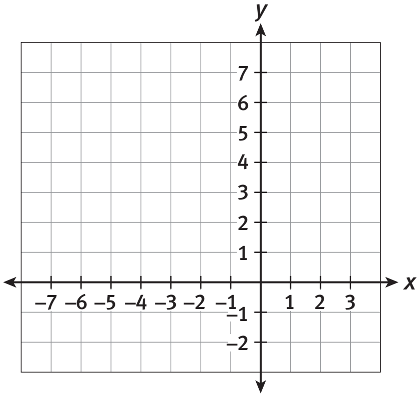
d. $f(x) = \left(\frac{1}{3}\right)^x$

$g(x) = \left(\frac{1}{3}\right)^x - 2$

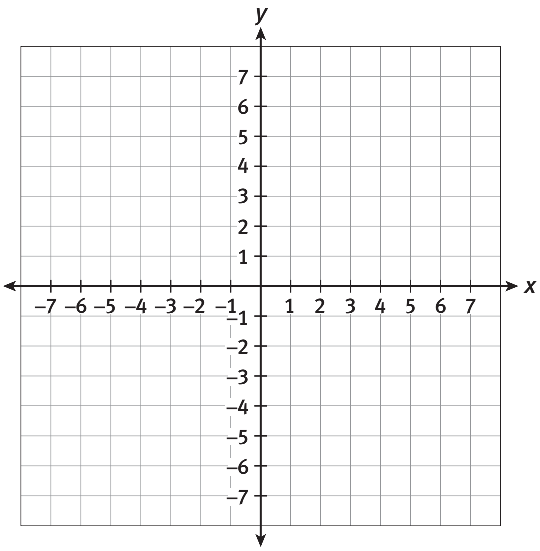
My Notes

3. Attend to precision. Describe how each function results from transforming a parent graph of the form $f(x) = b^x$. Then sketch the parent graph and the given function on the same axes. Give the domain and range of each function in interval notation. Give the equations of any asymptotes.

a. $g(x) = 3^{x+4} + 1$

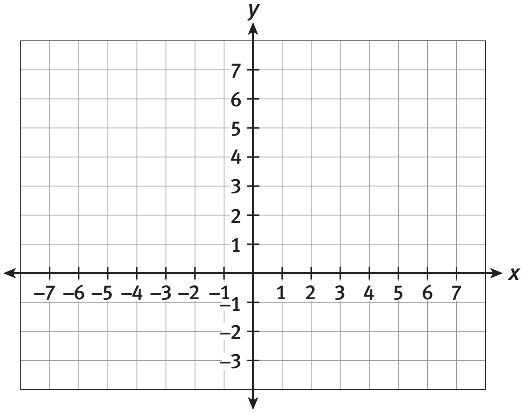


b. $g(x) = 2\left(\frac{1}{3}\right)^x - 4$

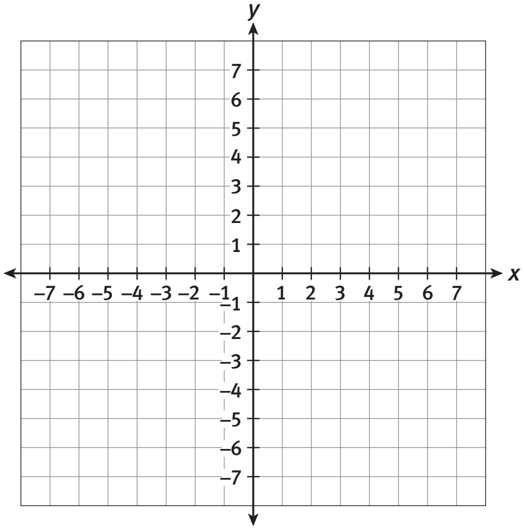


My Notes

c. $g(x) = \frac{1}{2}(4)^{x-4} - 2$



4. Describe how the function $g(x) = -3(2)^{x-6} + 5$ results from transforming a parent graph $f(x) = 2^x$. Sketch both graphs on the same axes. Give the domain and range of each function in interval notation. Give the equations of any asymptotes. Use a graphing calculator to check your work.



Check Your Understanding

5. **Reason quantitatively.** Explain how to change the equation of a parent graph $f(x) = 4^x$ to a translation that is left 6 units and a vertical shrink of 0.5.
6. Write the parent function $f(x)$ of $g(x) = -3(2)^{(x+2)} - 1$ and describe how the graph of $g(x)$ is a translation of the parent function.

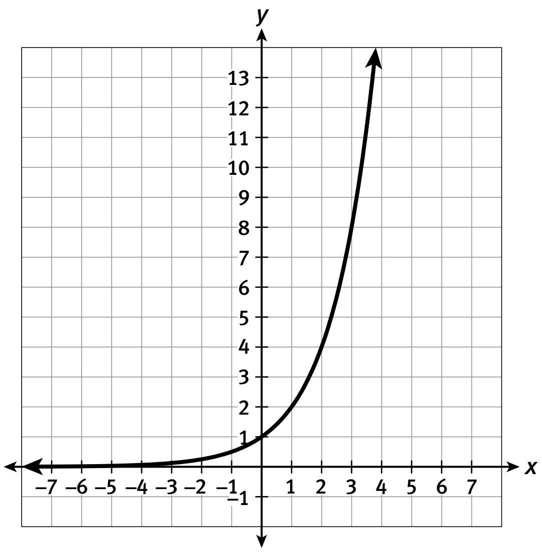
LESSON 21-4 PRACTICE

Describe how each function results from transforming a parent graph of the form $f(x) = b^x$. Then sketch the parent graph and the given function on the same axes. State the domain and range of each function and give the equations of any asymptotes.

7. $g(x) = 10^{x-2} - 3$
8. $g(x) = \frac{1}{2}(2)^{x-4} + 1$
9. $g(x) = -2\left(\frac{1}{3}\right)^{x+4}$

Make use of structure. Write the equation that indicates each transformation of the parent equation $f(x) = 2^x$. Then use the graph below and draw and label each transformation.

10. For $g(x)$, the y-intercept is at (0, 3).
11. For $h(x)$, the exponential growth factor is 0.5.
12. For $k(x)$, the graph of $f(x)$ is horizontally translated to the right 3 units.
13. For $l(x)$, the graph of $f(x)$ is vertically translated upward 2 units.



My Notes

MATH TIP

Exponential functions that describe examples of (continuous) exponential growth or decay use *e* for the base. You will learn more about the importance of *e* in Precalculus.

Learning Targets:

- Graph the function $f(x) = e^x$.
- Graph transformations of $f(x) = e^x$.

SUGGESTED LEARNING STRATEGIES: Quickwrite, Group Presentation, Debriefing

1. **Use appropriate tools strategically.** On a graphing calculator, set $Y_1 = x$ and $Y_2 = \left(1 + \frac{1}{x}\right)^x$. Let x increase by increments of 100. Describe what happens to the table of values for Y_2 as x increases.

This irrational constant is called *e* and is often used in exponential functions.

2. a. On a graphing calculator, enter $Y_1 = e^x$. Using the table of values associated with Y_1, complete the table below.

x	$Y_1 = e^x$
0	
1	
2	
3	

b. **Reason quantitatively.** Which row in the table gives the approximate value of *e*? Explain.

c. What kind of number does *e* represent?

3. a. Complete the table below.

x	x^{-1}	x^0	x^1	x^2	x^3
2	0.5	1	2	4	8
e					
3	0.3333	1	3	9	27

b. Graph the functions $f(x) = e^x$, $g(x) = 2^x$, and $h(x) = 3^x$ on the same coordinate plane.

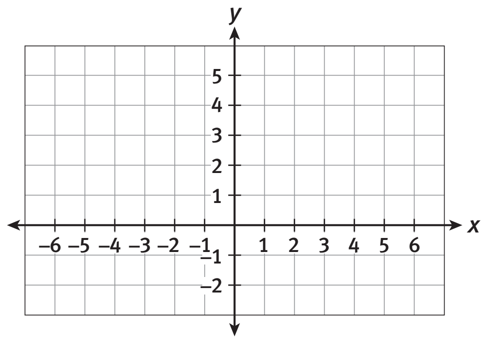
c. Compare $f(x)$ with $g(x)$ and $h(x)$. Which features are the same? Which are different?

My Notes

4. Graph the parent function $f(x) = e^x$ and the function $g(x)$ by applying the correct vertical stretch, shrink, reflection over the x-axis, or translation. Write a description for the transformation. State the domain and range of each function. Give the equation of any asymptotes.

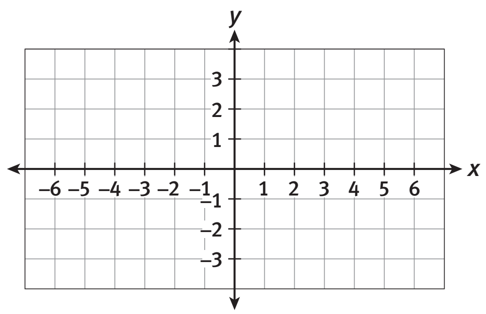
a. $f(x) = e^x$

$g(x) = -\frac{1}{2}(e^x)$

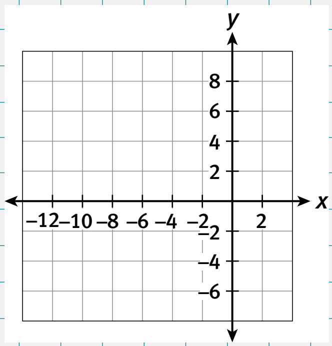


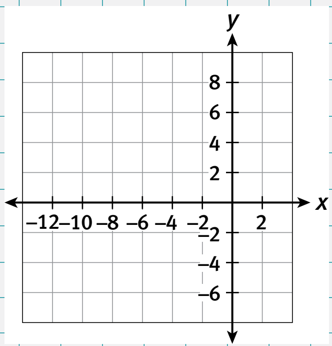
b. $f(x) = e^x$

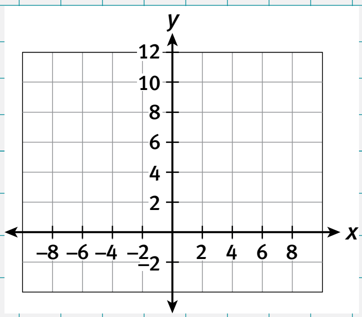
$g(x) = e^{x-4} + 1$



My Notes







5. Graph the parent graph *f* and the function *g* by applying the correct transformation. Write a description of each transformation. State the domain and range of each function. Give the equation of any asymptotes.

a. $f(x) = e^x$

$g(x) = 2e^x - 5$

b. $f(x) = e^x$

$g(x) = 2e^{x+1}$

c. $f(x) = e^x$

$g(x) = \frac{1}{2}(e^{x-4}) - 2$

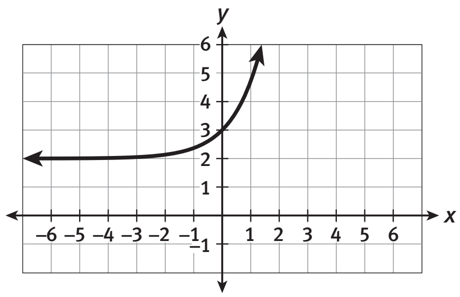
6. Explain how the parameters a, c, and d transform the parent graph $f(x) = b^x$ to produce the graph of the function $g(x) = a(b)^{x-c} + d$.

Check Your Understanding

Match each exponential expression with its graph.

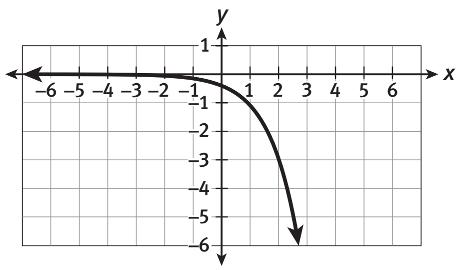
7. $f(x) = 3e^x$

A.



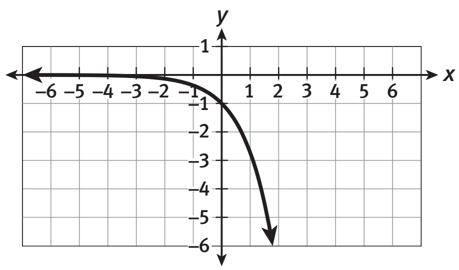
8. $f(x) = -0.4e^x$

B.



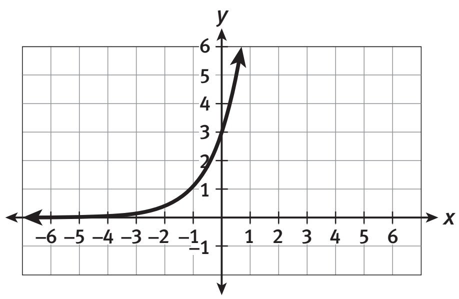
9. $f(x) = e^x + 2$

C.



10. $f(x) = -e^x$

D.



My Notes

LESSON 21-5 PRACTICE

Model with mathematics. Describe how each function results from transforming a parent graph of the form $f(x) = e^x$. Then sketch the parent graph and the given function on the same axes. State the domain and range of each function and give the equations of any asymptotes.

11. $g(x) = \frac{1}{4}e^x + 5$

12. $g(x) = e^{x-3} - 4$

13. $g(x) = -4e^{x-3} + 3$

14. $g(x) = 2e^{x+4}$

15. **Critique the reasoning of others.** On Cameron's math test, he was asked to describe the transformations from the graph of $f(x) = e^x$ to the graph of $g(x) = e^{x-2} - 2$. Cameron wrote "translation left 2 units and down 2 units." Do you agree or disagree with Cameron? Explain your reasoning.

16. What similarities, if any, are there between the functions studied in this lesson and the previous lesson?

ACTIVITY 21 PRACTICE

Write your answers on notebook paper.
Show your work.

Lesson 21-1

1. a. Complete the table so that the function represented is a linear function.

x	1	2	3	4	5
$f(x)$	5.4	6.7			10.6

b. What function is represented in the data?

2. a. How do you use a table of values to determine if the relationship of $y = 3x + 2$ is a linear relationship?
b. How do you use a graph to determine if the relationship in part a is linear?

3. Which relationship is nonlinear?
A. (2, 12), (5, 18), (6.5, 21)
B. $(6, x + 2)$, $(21, x + 7)$, $(-9, x - 3)$
C. (0.25, 1.25), (1.25, 2.50), (2.50, 5.00)
D. $(-5, 20)$, $(-3, 12)$, $(-1, 4)$

4. Determine if the table of data can be modeled by a linear function. If so, give an equation for the function. If not, explain why not.

x	0	1	2	3	4
y	$\frac{1}{5}$	$\frac{3}{5}$	1	$1\frac{2}{5}$	$1\frac{4}{5}$

5. Which relationship has the greatest value for $x = 4$?
A. $y = 5(3)^x + 2$
B. $y = 5(2^x + 3)$
C. $y = 5(3x + 2)$
D. $y = 5(2)^{x+3}$

6. Ida paints violets onto porcelain plates. She paints a spiral that is a sequence of violets, the size of each consecutive violet being a fraction of the size of the preceding violet. The table below shows the width of the first three violets in the continuing pattern.

Violet Number	1	2	3
Width (cm)	4	3.2	2.56

a. Is Ida's shrinking violet pattern an example of an exponential function? Explain.
b. Find the width of the fourth and fifth violets in the sequence.
c. Write an equation to express the size of the smallest violet in terms of the number of violets on the plate.
d. If a plate has a total of 10 violets, explain two different ways to determine the size of the smallest violet.

Lesson 21-2

7. Which statement is NOT true for the exponential function $f(x) = 4(0.75)^x$?
A. Exponential growth factor is 75%.
B. Percent of decrease is 25%.
C. The scale factor is 0.75.
D. The decay rate is 25%.

8. For the exponential function $f(x) = 3.2(1.5)^x$, identify the value of the parameters a and b. Then explain their meaning, using the vocabulary from the lesson.

9. Decide whether each table of data can be modeled by a linear function, an exponential function, or neither. If the data can be modeled by a linear or exponential function, give an equation for the function.

a.

x	0	1	2	3	4
y	24	18	12	6	0

b.

x	0	1	2	3	4
y	36	18	9	4.5	2.25

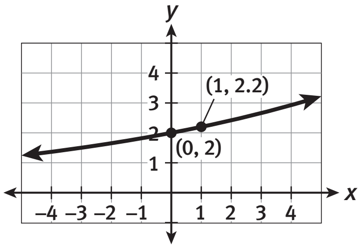
10. Sixteen teams play in a one-game elimination match. The winners of the first round go on to play a second round until only one team remains undefeated and is declared the champion.
a. Make a table of values for the number of rounds and the number of teams participating.
b. What is the reasonable domain and the range of this function? Explain.
c. Find the rate of decay.
d. Find the decay factor.

Lesson 21-3

11. Which of the following functions have the same graph?

A. $f(x) = \left(\frac{1}{4}\right)^x$
B. $f(x) = 4^x$
C. $f(x) = 4^{-x}$
D. $f(x) = x^4$

12. Which function is modeled in the graph below?



A. $y = (2)^x$
B. $y = 2(1.1)^x$
C. $y = (2)^{1.1x}$
D. $y = 2.1x$

13. For each exponential function, state the domain and range, whether the function increases or decreases, and the y-intercept.

a. $y = 2(4)^x$
b. $y = 3\left(\frac{1}{2}\right)^x$
c. $y = -(0.3)^x$
d. $y = -3(5.2)^x$

14. The *World Factbook* produced by the Central Intelligence Agency estimates the July 2012 United States population as 313,847,465. The following rates are also reported as estimates for 2012.

Birth rate: 13.7 births/1000 population

Death rate: 8.4 deaths/1000 population

Net migration rate: 3.62 migrant(s)/1000 population

a. Write a percent for each rate listed above.
b. Combine the percents from part a to find the overall growth rate for the United States.
c. The exponential growth factor for a population is equal to the growth rate plus 100%. What is the exponential growth rate for the United States?
d. Write a function to express the United States population as a function of years since 2012.
e. Use the function from part d to predict the United States population in the year 2050.

15. Under what conditions is the function $f(x) = a(3)^x$ increasing?

Lesson 21-4

16. Describe how each function results from transforming a parent graph of the form $f(x) = b^x$. Then sketch the parent graph and the given function on the same axes. State the domain and range of each function and give the equations of any asymptotes.

a. $g(x) = 2^{x+3} - 4$
b. $g(x) = -3\left(\frac{1}{2}\right)^x + 2$
c. $g(x) = \frac{1}{2}(3)^{x+3} - 4$

17. a. Explain why a change in c for the function $a(b)^{x-c} + d$ causes a horizontal translation.
b. Explain why a change in d for the function $a(b)^{x-c} + d$ causes a vertical translation.

18. Which transformation maps the graph of $f(x) = 3^x$ to $g(x) = \left(\frac{1}{3}\right)^x$?

A. horizontal translation
B. shrink
C. reflection
D. vertical translation

Lesson 21-5

19. Is $f(x) = e^x$ an increasing or a decreasing function? Explain your reasoning.

20. Which function has a y-intercept of (0, 0)?

A. $y = e^x + 1$
B. $y = -e^x + 1$
C. $y = e^x - 1$
D. $y = e^x$

21. What ordered pair do $f(x) = e^x$ and $g(x) = 2^x$ have in common?

MATHEMATICAL PRACTICES
Attend to Precision

22. Explain the difference between $y = x^2$ and $y = 2^x$.

Logarithms and Their Properties

ACTIVITY 22

Earthquakes and Richter Scale
Lesson 22-1 Exponential Data

My Notes

Learning Targets:

- Complete tables and plot points for exponential data.
- Write and graph an exponential function for a given context.
- Find the domain and range of an exponential function.

SUGGESTED LEARNING STRATEGIES: Summarizing, Paraphrasing, Create Representations, Quickwrite, Close Reading, Look for a Pattern

In 1935, Charles F. Richter developed the Richter magnitude test scale to compare the size of earthquakes. The Richter scale is based on the amplitude of the seismic waves recorded on seismographs at various locations after being adjusted for distance from the epicenter of the earthquake.

Richter assigned a magnitude of 0 to an earthquake whose amplitude on a seismograph is 1 micron, or 10^{-4} cm. According to the Richter scale, a magnitude 1.0 earthquake causes 10 times the ground motion of a magnitude 0 earthquake. A magnitude 2.0 earthquake causes 10 times the ground motion of a magnitude 1.0 earthquake. This pattern continues as the magnitude of the earthquake increases.

1. **Reason quantitatively.** How does the ground motion caused by earthquakes of these magnitudes compare?
   a. magnitude 5.0 earthquake compared to magnitude 4.0

   b. magnitude 4.0 earthquake compared to magnitude 1.0

   c. magnitude 4.0 earthquake compared to magnitude 0

The sign below describes the effects of earthquakes of different magnitudes. Read through this sign with your group and identify any words that might be unfamiliar. Find their meanings to aid your understanding.

Typical Effects of Earthquakes of Various Magnitudes

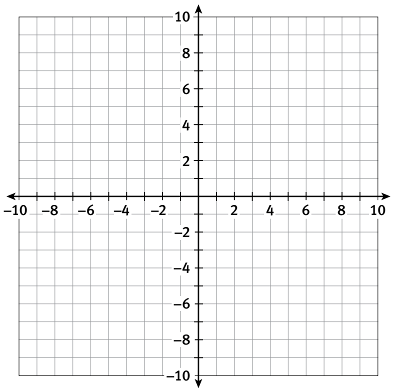
1.0	Very weak, no visible damage
2.0	Not felt by humans
3.0	Often felt, usually no damage
4.0	Windows rattle, indoor items shake
5.0	Damage to poorly constructed structures, slight damage to well-designed buildings
6.0	Destructive in populated areas
7.0	Serious damage over large geographic areas
8.0	Serious damage across areas of hundreds of miles
9.0	Serious damage across areas of hundreds of miles
10.0	Extremely rare, never recorded

My Notes

2. Complete the table to show how many times as great the ground motion is when caused by each earthquake as compared to a magnitude 0 earthquake.

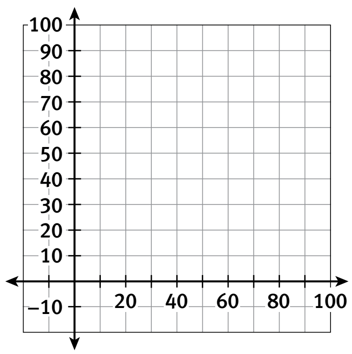
Magnitude	Ground Motion Compared to Magnitude 0
1.0	10
2.0	100
3.0	
4.0	
5.0	
6.0	
7.0	
8.0	
9.0	
10.0	

3. In parts a–c below, you will graph the data from Item 2. Let the horizontal axis represent the magnitude of the earthquake and the vertical axis represent the amount of ground motion caused by the earthquake as compared to a magnitude 0 earthquake. Alternatively, use technology to perform an exponential regression.



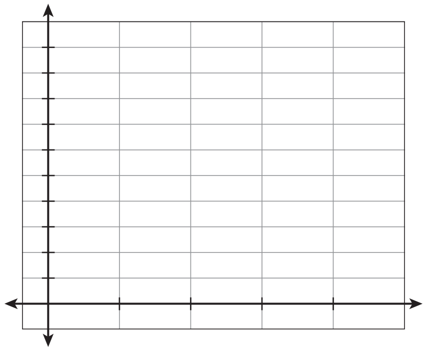
a. Plot the data using a grid that displays $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$. Explain why this grid is or is not a good choice.

b. Plot the data using a grid that displays $-10 \leq x \leq 100$ and $-10 \leq y \leq 100$. Explain why this grid is or is not a good choice.



My Notes

c. Scales may be easier to choose if only a subset of the data is graphed and if different scales are used for the horizontal and vertical axes. Determine an appropriate subset of the data and a scale for the graph. Plot the data and label and scale the axes. Draw a function that fits the plotted data.



d. Write a function $G(x)$ for the ground motion caused compared to a magnitude 0 earthquake by a magnitude x earthquake.

Check Your Understanding

4. What is the domain of the function in Item 3d? Is the graph of the function continuous?

5. Use the graph from Item 3c to estimate how many times greater the ground motion of an earthquake of magnitude 3.5 is than a magnitude 0 earthquake. Solve the equation you wrote in Item 3d to check that your estimate is reasonable.

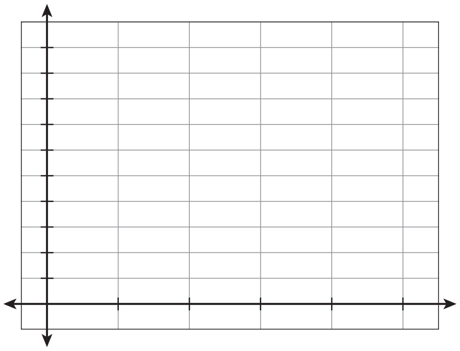
6. Make sense of problems. In Item 3, the data were plotted so that the ground motion caused by the earthquake was a function of the magnitude of the earthquake.

a. Is the ground motion a result of the magnitude of an earthquake, or is the magnitude of an earthquake the result of ground motion?

b. Based your answer to part a, would you choose ground motion or magnitude as the independent variable of a function relating the two quantities? What would you choose as the dependent variable?

My Notes

c. Make a new graph of the data plotted Item 3c so that the magnitude of the earthquake is a function of the ground motion caused by the earthquake. Scale the axes and draw a function that fits the plotted data.



7. Let the function you graphed in Item 6c be $y = M(x)$, where M is the magnitude of an earthquake for which there is x times as much ground motion as a magnitude 0 earthquake.

a. Identify a reasonable domain and range of the function $y = G(x)$ from Item 3d and the function $y = M(x)$ in this situation. Use interval notation.

	Domain	Range
$y = G(x)$	____________	____________
$y = M(x)$	____________	____________

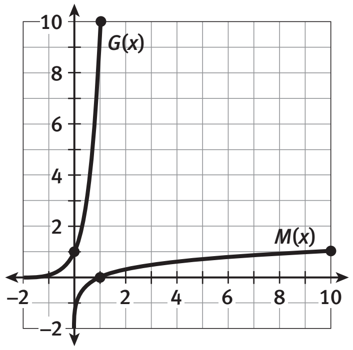
b. In terms of the problem situation, describe the meaning of an ordered pair on the graphs of $y = G(x)$ and $y = M(x)$.

$y = G(x)$ ______________________, ______________________

$y = M(x)$ ______________________, ______________________

My Notes

c. A portion of the graphs of $y = G(x)$ and $y = M(x)$ is shown on the same set of axes. Describe any patterns you observe.



Check Your Understanding

8. How did you choose the scale of the graph you drew in Item 6c?

9. What is the relationship between the functions *G* and *M*?

LESSON 22-1 PRACTICE

How does the ground motion caused by earthquakes of these magnitudes compare?

10. magnitude 5.0 compared to magnitude 2.0

11. magnitude 7.0 compared to magnitude 0

12. magnitude 6.0 compared to magnitude 5.0

13. A 1933 California earthquake had a Richter scale reading of 6.3. How many times more powerful was the Alaska 1964 earthquake with a reading of 8.3?

14. **Critique the reasoning of others.** Garrett said that the ground motion of an earthquake of magnitude 6 is twice the ground motion of an earthquake of magnitude 3. Is Garrett correct? Explain.

My Notes

Learning Targets:

- Use technology to graph $y = \log x$.
- Evaluate a logarithm using technology.
- Rewrite exponential equations as their corresponding logarithmic equations.
- Rewrite logarithmic equations as their corresponding exponential equations.

SUGGESTED LEARNING STRATEGIES: Close Reading, Vocabulary Organizer, Create Representations, Quickwrite, Think-Pair-Share

MATH TERMS

A **logarithm** is an exponent to which a base is raised that results in a specified value.

A **common logarithm** is a base 10 logarithm, such as $\log 100 = 2$, because $10^2 = 100$.

The Richter scale uses a base 10 ***logarithmic*** scale. A base 10 logarithmic scale means that when the ground motion is expressed as a power of 10, the magnitude of the earthquake is the exponent. You have seen this function $G(x) = 10^x$, where x is the magnitude, in Item 3d of the previous lesson.

The function M is the inverse of an exponential function G whose base is 10. The algebraic rule for M is a ***common logarithmic*** function. Write this function as $M(x) = \log x$, where x is the ground motion compared to a magnitude 0 earthquake.

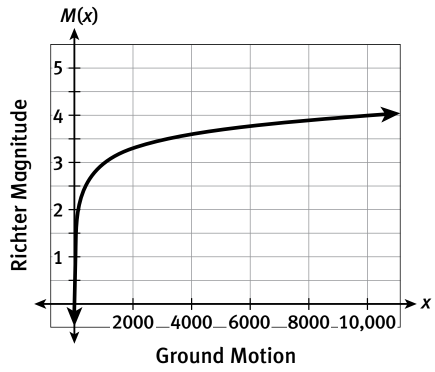
TECHNOLOGY TIP

The LOG key on your calculator is for common, or base 10, logarithms.

1. Graph $M(x) = \log x$ on a graphing calculator.

a. Make a sketch of the calculator graph. Be certain to label and scale each axis.

b. Use M to estimate the magnitude of an earthquake that causes 120,000 times the ground motion of a magnitude 0 earthquake. Describe what would happen if this earthquake were centered beneath a large city.



c. Use M to determine the amount of ground motion caused by the 2002 magnitude 7.9 Denali earthquake compared to a magnitude 0 earthquake.

2. Complete the tables below to show the relationship between the exponential function base 10 and its inverse, the common logarithmic function.

x	$y = 10^x$
0	$10^0 = 1$
1	
2	
3	
$\log x$	

x	$y = \log x$
$1 = 10^0$	$\log 1 = 0$
$10 = 10^1$	
$100 = 10^2$	
$1000 = 10^3$	
10^x	

MATH TIP

You can also write the equation $y = \log x$ as $y = \log_{10} x$. In the equation $y = \log x$, 10 is understood to be the base. Just as exponential functions can have bases other than 10, ***logarithmic functions*** can also be expressed with bases other than 10.

My Notes

MATH TIP

Recall that two functions are inverses when $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

The exponent x in the equation $y = 10^x$ is the common logarithm of y. This equation can be rewritten as $\log y = x$.

3. Use the information in Item 2 to write a logarithmic statement for each exponential statement.

a. $10^4 = 10{,}000$ **b.** $10^{-1} = \frac{1}{10}$

4. Use the information in Item 2 to write each logarithmic statement as an exponential statement.

a. $\log 100{,}000 = 5$ **b.** $\log\left(\frac{1}{100}\right) = -2$

5. Evaluate each logarithmic expression without using a calculator.

a. $\log 1000$ **b.** $\log \frac{1}{10{,}000}$

Check Your Understanding

6. What function has a graph that is symmetric to the graph of $y = \log x$ about the line $y = x$? Graph both functions and the line $y = x$.

7. Evaluate $\log 10^x$ for $x = 1, 2, 3,$ and 4.

8. Let $f(x) = 10^x$ and let $g(x) = f^{-1}(x)$. What is the algebraic rule for $g(x)$? Describe the relationship between $f(x)$ and $g(x)$.

LESSON 22-2 PRACTICE

9. Evaluate without using a calculator.

a. $\log 10^6$ **b.** $\log 1{,}000{,}000$ **c.** $\log \frac{1}{100}$

10. Write an exponential statement for each.

a. $\log 10 = 1$ **b.** $\log \frac{1}{1{,}000{,}000} = -6$ **c.** $\log a = b$

11. Write a logarithmic statement for each.

a. $10^7 = 10{,}000{,}000$ **b.** $10^0 = 1$ **c.** $10^m = n$

12. Model with mathematics. The number of decibels D of a sound is modeled with the equation $D = 10\log\left(\frac{I}{10^{-12}}\right)$ where I is the intensity of the sound measured in watts. Find the number of decibels in each of the following:

a. whisper with $I = 10^{-10}$

b. normal conversation with $I = 10^{-6}$

c. vacuum cleaner with $I = 10^{-4}$

d. front row of a rock concert with $I = 10^{-1}$

e. military jet takeoff with $I = 10^2$

My Notes

Learning Targets:

- Make conjectures about properties of logarithms.
- Write and apply the Product Property and Quotient Property of Logarithms.
- Rewrite logarithmic expressions by using properties.

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Create Representations, Look for a Pattern, Quickwrite, Guess and Check

You have already learned the properties of exponents. Logarithms also have properties.

1. Complete these three properties of exponents.

$a^m \cdot a^n =$ ________________

$\dfrac{a^m}{a^n} =$ ________________

$(a^m)^n =$ ________________

2. **Use appropriate tools strategically.** Use a calculator to complete the tables below. Round each answer to the nearest thousandth.

x	$y = \log x$
1	0
2	
3	
4	
5	

x	$y = \log x$
6	
7	
8	
9	
10	

3. Add the logarithms from the tables in Item 2 to see if you can develop a property. Find each sum and round each answer to the nearest thousandth.

$\log 2 + \log 3 =$ ________________

$\log 2 + \log 4 =$ ________________

$\log 2 + \log 5 =$ ________________

$\log 3 + \log 3 =$ ________________

My Notes

4. Compare the answers in Item 3 to the tables of data in Item 2.

a. Express regularity in repeated reasoning. Is there a pattern or property when these logarithms are added? If yes, explain the pattern that you have found.

b. State the property of logarithms that you found by completing the following statement.

$\log m + \log n =$ ________________

5. Explain the connection between the property of logarithms stated in Item 4 and the corresponding property of exponents in Item 1.

6. Graph $y_1 = \log 2 + \log x$ and $y_2 = \log 2x$ on a graphing calculator. What do you observe? Explain.

TECHNOLOGY TIP

When using the LOG key on a graphing calculator, a leading parenthesis is automatically inserted. The closing parenthesis for logarithmic expressions must be entered manually. So entering $\log 2 + \log x$ without closing the parenthesis that the calculator will place before the 2 will NOT give the correct result.

Check Your Understanding

Identify each statement as true or false. Justify your answers.

7. $\log mn = (\log m)(\log n)$

8. $\log xy = \log x + \log y$

9. Make a conjecture about the property of logarithms that relates to the property of exponential equations that states the following:

$\frac{a^m}{a^n} = a^{m-n}$.

10. Use the information from the tables in Item 2 to provide examples that support your conjecture in Item 9.

11. Graph $y_1 = \log x - \log 2$ and $y_2 = \log \frac{x}{2}$ on a graphing calculator. What do you observe?

My Notes

Check Your Understanding

Use the information from the tables in Item 2 and the properties in Items 4b and 9.

12. Write two different logarithmic expressions to find a value for log 36.

13. Write a logarithmic expression that contains a quotient and simplifies to 0.301.

14. **Construct viable arguments.** Show that $\log(3 + 4) \neq \log 3 + \log 4$.

LESSON 22-3 PRACTICE

Use the table of logarithmic values at the beginning of the lesson to evaluate the logarithms in Items 15 and 16. Do not use a calculator.

15. **a.** $\log\left(\frac{8}{3}\right)$

b. log 24

c. log 64

d. log 27

16. **a.** $\log\left(\frac{4}{9}\right)$

b. log 2.25

c. log 144

d. log 81

17. Rewrite $\log 7 + \log x - (\log 3 + \log y)$ as a single logarithm.

18. Rewrite $\log\left(\frac{8m}{9n}\right)$ as a sum of four logarithmic terms.

19. **Make use of structure.** Rewrite $\log 8 + \log 2 - \log 4$ as a single logarithm and evaluate the result using the table at the beginning of the lesson.

My Notes

Learning Targets:

- Make conjectures about properties of logarithms.
- Write and apply the Power Property of Logarithms.
- Rewrite logarithmic expressions by using their properties.

SUGGESTED LEARNING STRATEGIES: Think-Pair-Share, Create Representations

1. Make a conjecture about the property of logarithms that relates to the property of exponents that states the following: $(a^m)^n = a^{mn}$.

2. Use the information from the tables in Item 2 in the previous lesson and the properties developed in Items 4 and 9 in the previous lesson to support your conjecture in Item 1.

3. Use appropriate tools strategically. Graph $y_1 = 2 \log x$ and $y_2 = \log x^2$ on a graphing calculator. What do you observe?

Check Your Understanding

Identify each statement as true or false. Justify your answer.

4. $2 \log \sqrt{m} = \log m$

5. $\log 10^2 = \log 2^{10}$

6. Express regularity in repeated reasoning. The logarithmic properties that you conjectured and then verified in this lesson and the previous lesson are listed below. State each property.

Product Property: ______________________________

Quotient Property: ______________________________

Power Property: ______________________________

My Notes

7. Use the properties from Item 6 to rewrite each expression as a single logarithm. Assume all variables are positive.
a. $\log x - \log 7$

b. $2 \log x + \log y$

8. Use the properties from Item 6 to expand each expression. Assume all variables are positive.
a. $\log 5xy^4$

b. $\log \frac{x}{y^3}$

9. Rewrite each expression as a single logarithm. Then evaluate.
a. $\log 2 + \log 5$

b. $\log 5000 - \log 5$

c. $2 \log 5 + \log 4$

Check Your Understanding

10. Explain why $\log (a + 10)$ does not equal $\log a + 1$.

11. Explain why $\log (-100)$ is not defined.

LESSON 22-4 PRACTICE

Attend to precision. Rewrite each expression as a single logarithm. Then evaluate the expression without using a calculator.

12. $\log 5 + \log 20$

13. $\log 3 - \log 30$

14. $2 \log 400 - \log 16$

15. $\log \frac{1}{400} + 2 \log 2$

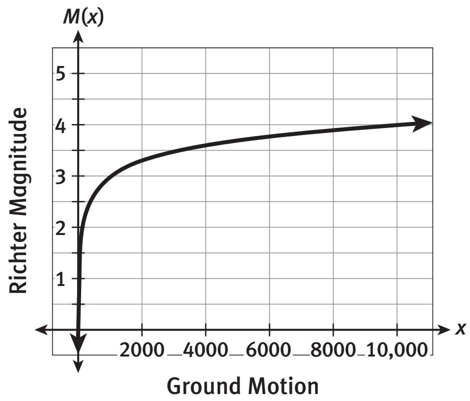
16. $\log 100 + \log\left(\frac{1}{100}\right)$.

17. Expand the expression $\log bc^3d^2$.

ACTIVITY 22 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 22-1



1. What is the y-intercept of the graph?

2. What is the x-intercept of the graph?

3. Is $M(x)$ an increasing or decreasing function?

4. Which of these statements are NOT true regarding the graph above?
A. The graph contains the point (1, 0).
B. The graph contains the point (10, 1).
C. The domain is $x > 0$.
D. The x-axis is an asymptote.

Lesson 22-2

5. Use a calculator to find a decimal approximation rounded to three decimal places.
a. $\log 47$
b. $\log 32.013$
c. $\log\left(\frac{5}{7}\right)$
d. $\log -20$

6. A logarithm is a(n)
A. variable.
B. constant.
C. exponent.
D. coefficient.

7. Write an exponential statement for each logarithmic statement below.
a. $\log 10{,}000 = 4$
b. $\log \frac{1}{1{,}000{,}000{,}000} = -9$
c. $\log a = 6$

8. Write a logarithmic statement for each exponential statement below.
a. $10^{-2} = \frac{1}{100}$
b. $10^1 = 10$
c. $10^4 = n$

9. Evaluate without using a calculator.
a. $\log 10^5$
b. $\log 100$
c. $\log \frac{1}{100{,}000}$

10. If $\log a = x$, and $10 < a < 100$, what values are acceptable for x?
A. $0 < x < 1$
B. $1 < x < 2$
C. $2 < x < 3$
D. $10 < x < 100$

Lesson 22-3

11. If $\log 2 = 0.301$ and $\log 3 = 0.447$, find each of the following using only these values and the properties of logarithms. Show your work.
a. $\log 6$
b. $\log\left(\frac{2}{3}\right)$
c. $\log 1.5$
d. $\log 18$

12. Which expression does NOT equal 3?

A. $\log 10^3$

B. $\dfrac{\log 10^5}{\log 10^2}$

C. $\log\left(\dfrac{10^7}{10^4}\right)$

D. $\log 10^4 - \log 10$

13. Explain the connection between the exponential equation ($10^3 \cdot 10^5 = 10^8$) and the logarithmic equation ($\log 10^3 + \log 10^5 = \log 10^8$).

14. Rewrite each expression as a single logarithm.

a. $\log 2 + \log x - (\log 3 + \log y)$

b. $\log 5 - \log 7$

c. $(\log 24 + \log 12) - \log 6$

15. Expand each expression.

a. $\log\left(\dfrac{3x}{8y}\right)$

b. $\log\left(\dfrac{m+v}{3}\right)$

c. $\log\left(\dfrac{4}{9-u}\right)$

16. If $\log 2 = 0.301$ and $\log 3 = 0.477$, find each of the following using the properties of logarithms.

a. $\log 4$

b. $\log 27$

c. $\log \sqrt{2}$

d. $\log \sqrt{12}$

Lesson 22-4

17. Complete each statement to illustrate a property for logarithms.

a. Product Property $\quad \log uv = ?$

b. Quotient Property $\quad \log \dfrac{u}{v} = ?$

c. Power Property $\quad \log u^v = ?$

18. Rewrite each expression as a single logarithm. Then evaluate without using a calculator.

a. $\log 500 + \log 2$

b. $2\log 3 + \log \frac{1}{9}$

c. $\log 80 - 3\log 2$

19. Expand each expression.

a. $\log xy^2$

b. $\log \dfrac{xy}{z}$

c. $\log a^3b^2$

20. If $\log 8 = 0.903$ and $\log 3 = 0.477$, find each of the following using the properties of logarithms.

a. $\log 3^8$

b. $\log (2^3)^3$

c. $\log 8(3^2)$

21. Write each expression without using exponents.

a. $m \log n + \log n^m$

b. $\log (mn)^0$

c. $\log 2^4 + \log 2^3$

22. Which of the following statements is TRUE?

A. $\log \dfrac{x}{y} = \dfrac{\log x}{\log y}$

B. $\log \dfrac{x}{y} = y \log x$

C. $\log (x + y) = \log x + \log y$

D. $\log \sqrt{x} = \frac{1}{2} \log x$

MATHEMATICAL PRACTICES
Reason Abstractly and Quantitatively

23. Verify using the properties of logarithms that $\log 10^x - \log 10^4 = x - 4$. Then evaluate for $x = \pi$, using 3.14 for π.

Exponential Functions and Common Logarithms

WHETHER OR NOT

1. **Reason quantitatively.** Tell whether or not each table contains data that can be modeled by an exponential function. Provide an equation to show the relationship between x and y for the sets of data that are exponential.

   a.

x	0	1	2	3
y	3	6	12	24

   b.

x	0	1	2	3
y	2	4	6	8

   c.

x	0	1	2	3
y	108	36	12	4

2. Tell whether or not each function is increasing. State *increasing* or *decreasing*, and give the domain, range, and y-intercept of the function.

   a. $y = 4\left(\frac{2}{3}\right)^x$ **b.** $y = -3(4)^x$

3. Let $g(x) = 2(4)^{x+3} - 5$.
   a. Describe the function as a transformation of $f(x) = 4^x$.
   b. Graph the function using your knowledge of transformations.
   c. What is the horizontal asymptote of the graph of g?

4. Rewrite each exponential equation as a common logarithmic equation.

   a. $10^3 = 1000$ **b.** $10^{-4} = \frac{1}{10{,}000}$ **c.** $10^7 = 10{,}000{,}000$

5. **Make use of structure.** Rewrite each common logarithmic equation as an exponential equation.

   a. $\log 100 = 2$ **b.** $\log 100{,}000 = 5$ **c.** $\log \frac{1}{100{,}000} = -5$

6. Evaluate each expression without using a calculator.
   a. $\log 1000$ **b.** $\log 1$ **c.** $\log 2 + \log 50$

7. Evaluate using a calculator. Then rewrite each expression as a single logarithm without exponents and evaluate again as a check.
   a. $\log 5 + \log 3$ **b.** $\log 3^4$ **c.** $\log 3 - \log 9$

Exponential Functions and Common Logarithms

WHETHER OR NOT

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
	The solution demonstrates these characteristics:			
Mathematics Knowledge and Thinking (Items 1, 2, 3c, 4–7)	• Clear and accurate understanding of how to determine whether a table of data represents an exponential function • Clear and accurate understanding of the features of exponential functions and their graphs including domain and range • Fluency in evaluating and rewriting exponential and logarithmic equations and expressions	• Largely correct understanding of how to determine whether a table of data represents an exponential function • Largely correct understanding of the features of exponential functions and their graphs including domain and range • Little difficulty when evaluating and rewriting exponential and logarithmic equations and expressions	• Partial understanding of how to determine whether a table of data represents an exponential function • Partial understanding of the features of exponential functions and their graphs including domain and range • Some difficulty when evaluating and rewriting logarithmic and exponential equations and expressions	• Little or no understanding of how to determine whether a table of data represents an exponential function • Inaccurate or incomplete understanding of the features of exponential functions and their graphs including domain and range • Significant difficulty when evaluating and rewriting logarithmic and exponential equations and expressions
Problem Solving (Item 1)	• An appropriate and efficient strategy that results in a correct answer	• A strategy that may include unnecessary steps but results in a correct answer	• A strategy that results in some incorrect answers	• No clear strategy when solving problems
Mathematical Modeling / Representations (Items 1, 3b)	• Fluency in recognizing exponential data and modeling it with an equation • Effective understanding of how to graph an exponential function using transformations	• Little difficulty in accurately recognizing exponential data and modeling it with an equation • Largely correct understanding of how to graph an exponential function using transformations	• Some difficulty with recognizing exponential data and modeling it with an equation • Partial understanding of how to graph an exponential function using transformations	• Significant difficulty with recognizing exponential data and model it with an equation • Mostly inaccurate or incomplete understanding of how to graph an exponential function using transformations
Reasoning and Communication (Items 1a, 3a)	• Clear and accurate justification of whether or not data represented an exponential model • Precise use of appropriate math terms and language to describe a function as a transformation of another function	• Adequate justification of whether or not data represented an exponential model • Adequate and correct description of a function as a transformation of another function	• Misleading or confusing justification of whether or not data represented an exponential model • Misleading or confusing description of a function as a transformation of another function	• Incomplete or inadequate justification of whether or not data represented an exponential model • Incomplete or mostly inaccurate description of a function as a transformation of another function

Inverse Functions: Exponential and Logarithmic Functions

Undoing It All

Lesson 23-1 Logarithms in Other Bases

My Notes

Learning Targets:

- Use composition to verify two functions as inverse.
- Define the logarithm of y with base b.
- Write the Inverse Properties for logarithms.

SUGGESTED LEARNING STRATEGIES: Close Reading, Create Representations

In the first unit, you studied inverses of linear functions. Recall that two functions f and g are *inverses* of each other if and only if $f(g(x)) = x$ for all x in the domain of g, and $g(f(x)) = x$ for all x in the domain of f.

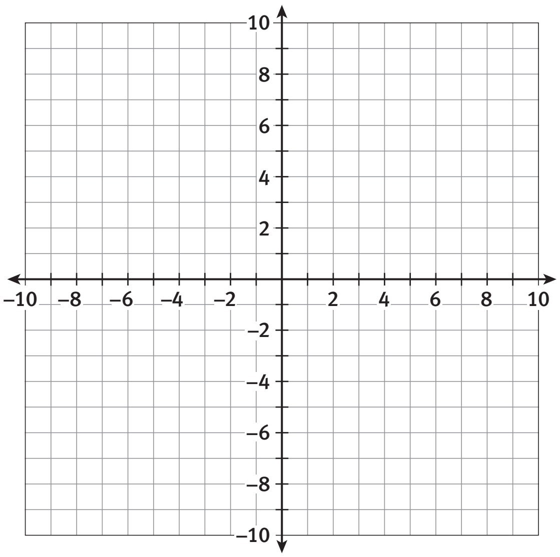
MATH TIP

To find the inverse of a function algebraically, interchange the x and y variables and then solve for y.

1. Find the inverse function $g(x)$ of the function $f(x) = 2x + 1$. Show your work.

2. Use the definition of inverse functions to prove that $f(x) = 2x + 1$ and the $g(x)$ function you found in Item 1 are inverse functions.

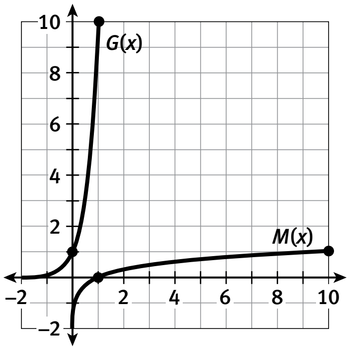
3. Graph $f(x) = 2x + 1$ and its inverse $g(x)$ on the grid below. What is the line of symmetry between the graphs?



In a previous activity, you investigated exponential functions with a base of 10 and their inverse functions, the common logarithmic functions. Recall in the Richter scale situation that $G(x) = 10^x$, where x is the magnitude of an earthquake. The inverse function is $M(x) = \log x$, where x is the ground motion compared to a magnitude 0 earthquake.

My Notes

4. A part of each of the graphs of $y = G(x)$ and $y = M(x)$ is shown below. What is the line of symmetry between the graphs? How does that line compare with the line of symmetry in Item 3?



Logarithms with bases other than 10 have the same properties as common logarithms.

The logarithm of y with base b, where $y > 0$, $b > 0$, $b \neq 1$, is defined as: $\log_b y = x$ if and only if $y = b^x$.

The exponential function $y = b^x$ and the logarithmic function $y = \log_b x$, where $b > 0$ and $b \neq 1$, are inverse functions. The (restricted) domain of one function is the (restricted) range of the other function. Likewise, the (restricted) range of one function is the (restricted) domain of the other function.

MATH TIP

The notation f^{-1} is used to indicate the inverse of the function f.

5. Let $g(x) = f^{-1}(x)$, the inverse of function f. Write the rule for g for each function f given below.

a. $f(x) = 5^x$ **b.** $f(x) = \log_4 x$ **c.** $f(x) = \log_e x$

Logarithms with base e are called ***natural logarithms***, and "$\log_e$" is written **ln**. So, $\log_e x$ is written $\ln x$.

6. Use the functions from Item 5. Complete the expression for each composition.

a. $f(x) = 5^x$

$f(g(x)) =$ __________ $= x$ $g(f(x)) =$ __________ $= x$

b. $f(x) = \log_4 x$

$f(g(x)) =$ __________ $= x$ $g(f(x)) =$ __________ $= x$

c. $f(x) = e^x$

$f(g(x)) =$ __________ $= x$ $g(f(x)) =$ __________ $= x$

My Notes

7. Use what you learned in Item 6 to complete these *Inverse Properties of Logarithms*. Assume $b > 0$ and $b \neq 1$.

a. $b^{\log_b x} =$ __________ **b.** $\log_b b^x =$ __________

8. Simplify each expression.

a. $6^{\log_6 x}$ **b.** $\log_3 3^x$

c. $7^{\log_7 x}$ **d.** $\log 10^x$

e. $\ln e^x$ **f.** $e^{\ln x}$

Check Your Understanding

9. Describe the process you use to find the inverse function $g(x)$ if $f(x) = 7x + 8$.

10. **Construct viable arguments.** Look at the graphs in Items 3 and 4. What can you conclude about the line of symmetry for a function and its inverse?

11. Answer each of the following as true or false. If false, explain your reasoning.

a. The "-1" in function notation f^{-1} means $\frac{1}{f}$.

b. Exponential functions are the inverse of logarithmic functions.

c. If the inverse is a function, then the original must be a function.

LESSON 23-1 PRACTICE

Let $g(x) = f^{-1}(x)$, the inverse of function f. Write the rule for g for each function f given below.

12. $f(x) = 3x - 8$

13. $f(x) = \frac{1}{2}x + 5$

14. $f(x) = 5x - 6$

15. $f(x) = -x + 7$

16. $f(x) = 7^x$

17. $f(x) = e^x$

18. $f(x) = \log_{12} x$

19. $f(x) = \ln x$

Simplify each expression.

20. $\log_9 9^x$

21. $15^{\log_{15} x}$

22. $\ln e^x$

23. $8^{\log_8 x}$

My Notes

Learning Targets:

- Apply the properties of logarithms in any base.
- Compare and expand logarithmic expressions.
- Use the Change of Base Formula.

SUGGESTED LEARNING STRATEGIES: Create Representations, Close Reading

When rewriting expressions in exponential and logarithmic form, it is helpful to remember that a *logarithm is an exponent*. The exponential statement $2^3 = 8$ is equivalent to the logarithmic statement $\log_2 8 = 3$.

Notice that the logarithmic expression is equal to 3, which is the exponent in the exponential expression.

1. Express each exponential statement as a logarithmic statement.

a. $3^4 = 81$ **b.** $6^{-2} = \frac{1}{36}$ **c.** $e^0 = 1$

2. Express each logarithmic statement as an exponential statement.

a. $\log_4 16 = 2$ **b.** $\log_5 125 = 3$ **c.** $\ln 1 = 0$

3. Evaluate each expression without using a calculator.

a. $\log_2 32$ **b.** $\log_4 \left(\frac{1}{64}\right)$

c. $\log_3 27$ **d.** $\log_{12} 1$

MATH TIP

Remember that a logarithm is an exponent. To evaluate the expression $\log_6 36$, find the exponent for 6 that gives the value 36. $6^2 = 36$. Therefore, $\log_6 36 = 2$.

Check Your Understanding

4. Why is the value of $\log_{-2} 16$ undefined?

5. Critique the reasoning of others. Mike said that the $\log_3$ of $\frac{1}{9}$ is undefined, because $3^{-2} = \frac{1}{9}$, and a log cannot have a negative value. Is Mike right? Why or why not?

The *Product, Quotient,* and *Power Properties* of common logarithms also extend to bases other than base 10.

6. Use the given property to rewrite each expression as a single logarithm. Then evaluate each logarithm in the equation to see that both sides of the equation are equal.

a. Product Property: $\log_2 4 + \log_2 8 =$ ___________

___________ + ___________ =

b. Quotient Property: $\log_3 27 - \log_3 3 =$ ___________

___________ − ___________ = ___________

c. Power Property: $2 \log_5 25 =$ ___________

2 • ___________ = ___________

My Notes

7. Expand each expression. Assume all variables are positive.

a. $\log_7\left(\frac{x}{y^3}\right)$

b. $\log_4 x^2y$

c. $\ln\left(\frac{x^2}{y^3}\right)$

8. Assume that x is any real number, and decide whether the statement is *always true, sometimes true,* or *never true*. If the statement is sometimes true, give the conditions for which it is true.

a. $\log 7 - \log 5 = \frac{\log 7}{\log 5}$

b. $\log_5 5^x = x$

c. $2^{\log_2 x^2} = x^2$

d. $\log_4 3 + \log_4 5 - \log_4 x = \log_4 15$

e. $2 \ln x = \ln x + \ln x$

Check Your Understanding

9. Attend to precision. Why is it important to specify the value of the variables as positive when using the Product, Quotient, and Power Properties of logarithms? Use Item 7 to state an example.

10. Simplify the following expression:
$\log 7 - \log 5$

Sometimes it is useful to change the base of a logarithmic expression. For example, the log key on a calculator is for common, or base 10, logs. Changing the base of a logarithm to 10 makes it easier to work with logarithms on a calculator.

11. Use the common logarithm function on a calculator to find the numerical value of each expression. Write the value in the first column of the table. Then write the numerical value using logarithms in base 2 in the second column.

	Numerical Value	$\log_2 a$
$\frac{\log 2}{\log 2}$	1	$\log_2 2$
$\frac{\log 4}{\log 2}$		
$\frac{\log 8}{\log 2}$		
$\frac{\log 16}{\log 2}$		
$\frac{\log N}{\log 2}$		

My Notes

12. The patterns observed in the table in Item 11 illustrate the ***Change of Base Formula***. Make a conjecture about the Change of Base Formula of logarithms.

$$\log_b x = \_\_\_\_\_\_\_\_\_\_$$

13. Consider the expression $\log_2 12$.

a. The value of $\log_2 12$ lies between which two integers?

b. Write an equivalent common logarithm expression for $\log_2 12$, using the Change of Base Formula.

c. Use a calculator to find the value of $\log_2 12$ to three decimal places. Compare the value to your answer from part a.

Check Your Understanding

14. Change each expression to a logarithmic expression in base 10. Use a calculator to find the value to three decimal places.

a. $\log_5 32$ **b.** $\log_3 104$

15. In Item 13, how do you find out which values the value of $\log_2 12$ lies between?

LESSON 23-2 PRACTICE

Write a logarithmic statement for each exponential statement.

16. $7^3 = 343$

17. $3^{-2} = \frac{1}{9}$

18. $e^m = u$

Write an exponential statement for each logarithmic statement.

19. $\log_6 1296 = 4$

20. $\log_{\frac{1}{2}} 4 = -2$

21. $\ln x = t$

Evaluate each expression without using a calculator.

22. $\log_4 64$

23. $\log_2\left(\frac{1}{32}\right)$

Change each expression to a logarithmic expression in base 10. Use a calculator to find the value to three decimal places.

24. $\log_3 7$

25. $\log_2 18$

26. $\log_{25} 4$

My Notes

Learning Targets:

- Find intercepts and asymptotes of logarithmic functions.
- Determine the domain and range of a logarithmic function.
- Write and graph transformations of logarithmic functions.

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Close Reading, Quickwrite

1. Examine the function $f(x) = 2^x$ and its inverse, $g(x) = \log_2 x$.

a. Complete the table of data for $f(x) = 2^x$. Then use that data to complete a table of values for $g(x) = \log_2 x$.

x	$f(x) = 2^x$
−2	
−1	
0	
1	
2	

x	$g(x) = \log_2 x$

b. Graph both $f(x) = 2^x$ and $g(x) = \log_2 x$ on the same grid.

c. What are the x- and y-intercepts for $f(x) = 2^x$ and $g(x) = \log_2 (x)$?

d. What is the line of symmetry between the graphs of $f(x) = 2^x$ and $g(x) = \log_2 x$?

e. State the domain and range of each function using interval notation.

f. What is the end behavior of the graph of $f(x) = 2^x$?

g. What is the end behavior of the graph of $g(x) = \log_2 (x)$?

My Notes

TECHNOLOGY TIP

The LN key on your calculator is the natural logarithm key.

h. Write the equation of any asymptotes of each function.

$f(x) = 2^x$ ______________________

$g(x) = \log_2 x$ ______________________

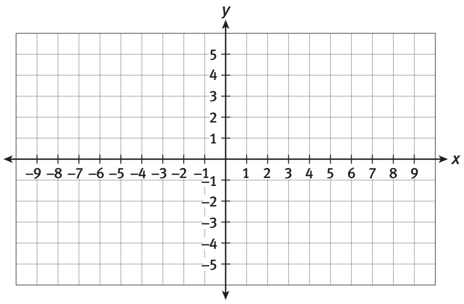
2. Examine the function $f(x) = e^x$ and its inverse, $g(x) = \ln x$.

a. Complete the table of data for $f(x) = e^x$. Then use those data to complete a table of values for $g(x) = \ln x$.

x	$f(x) = e^x$
−2	
−1	
0	
1	
2	

x	$g(x) = \ln x$

b. Graph both $f(x) = e^x$ and $g(x) = \ln x$ on the same grid.



c. What are the x- and y-intercepts for $f(x) = e^x$ and $g(x) = \ln x$?

d. What is the line of symmetry between the graphs of $f(x) = e^x$ and $g(x) = \ln x$?

e. State the domain and range of each function using interval notation.

f. What is the end behavior of the graph of $f(x) = e^x$?

g. What is the end behavior of the graph of $g(x) = \ln x$?

h. Write the equation of any asymptotes of each function.

$f(x) = e^x$______________________

$g(x) = \ln x$______________________

My Notes

Check Your Understanding

3. Make sense of problems. From the graphs you drew for Items 1 and 2, draw conclusions about the behavior of inverse functions with respect to:
a. the intercepts
b. the end behavior
c. the asymptotes

4. If a function has an intercept of (0, 0), what point, if any, will be an intercept for the inverse function?

Transformations of the graph of the function $f(x) = \log_b x$ can be used to graph functions of the form $g(x) = a \log_b (x - c) + d$, where $b > 0$, $b \neq 1$. You can draw a quick sketch of each parent graph, $f(x) = \log_b x$, by plotting the points $\left(\frac{1}{b}, -1\right)$, (1, 0), and $(b, 1)$.

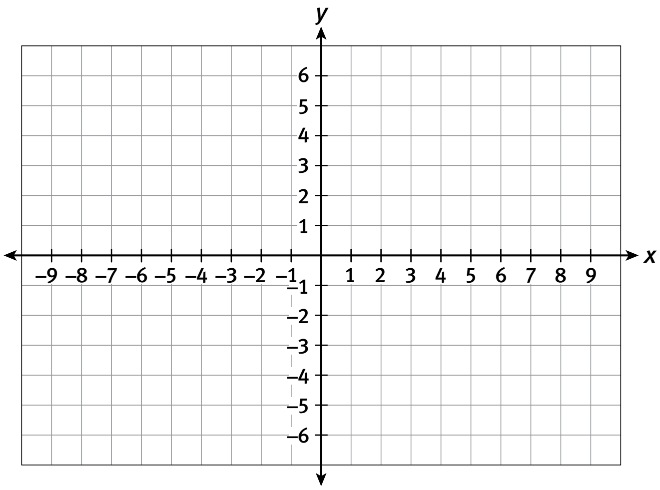
5. Sketch the parent graph $f(x) = \log_2 x$ on the axes below. Then, for each transformation of f, provide a verbal description and sketch the graph, including asymptotes.
a. $g(x) = 3 \log_2 x$

b. $h(x) = 3 \log_2 (x + 4)$

c. $j(x) = 3 \log_2 (x + 4) - 2$

d. $k(x) = \log_2 (8x)$

e. $m(x) = -3 \log_2 x$



6. Explain how the function $j(x) = 3 \log_2 (x + 4) - 2$ can be entered on a graphing calculator using the common logarithm key. Then graph the function on a calculator and compare the graph to your answer in Item 5c.

MATH TIP

Recall that a graph of the exponential function $f(x) = b^x$ can be drawn by plotting the points $\left(-1, \frac{1}{b}\right)$, (0, 1), and (1, b).

Switching the x- and y-coordinates of these points gives you three points on the graph of the inverse of $f(x) = b^x$, which is $f(x) = \log_b x$.

My Notes

7. Consider how the parameters a, c and d transform the graph of the general logarithmic function $y = a\log_b(x - c) + d$.

a. Use a graphing calculator to graph the parent function $f(x) = \log x$. Then, for each transformation of f, provide a verbal description of the transformation and the equation of the asymptote.

i. $y = -\log x$
ii. $y = 2\log x$
iii. $y = \log(x + 1)$
iv. $y = -3\log(x - 2) + 1$
v. $y = \frac{1}{2}\log x - 3$

b. Use a graphing calculator to graph the parent graph $f(x) = \ln x$. Then, for each transformation of f, provide a verbal description of the transformation and the equation of the asymptote.

i. $y = -\frac{1}{2}\ln(x + 1)$
ii. $y = 2\ln x + 1$
iii. $y = 3\ln(x - 1)$
iv. $y = -\ln x - 2$

c. Explain how the parameters a, c, and d transform the parent graph $f(x) = \log_b x$ to produce a graph of the function $g(x) = a\log_b(x - c) + d$.

Check Your Understanding

8. Look for and make use of structure.

a. Compare the effect of a in a logarithmic function $a\log_b x$ to a in a quadratic function ax^2 (assume a is positive).

b. Compare the effect of c in a logarithmic function $\log_b(x - c)$ to c in a quadratic function $(x - c)^2$.

LESSON 23-3 PRACTICE

9. Given an exponential function that has a y-intercept of 1 and no x-intercept, what is true about the intercepts of the function's inverse?

10. Make sense of problems. The inverse of a function has a domain of $(-\infty, \infty)$ and a range of $(0, \infty)$. What is true about the original function's domain and range?

Model with mathematics. Graph each function, using a parent graph and the appropriate transformations. Describe the transformations.

11. $f(x) = 2\log_2(x) - 6$

12. $f(x) = \log(x - 5) + 1$

13. $f(x) = \frac{1}{2}\ln x$

14. $f(x) = \log_2(x + 4) - 3$

15. $f(x) = 2\log(x - 1)$

16. $f(x) = -\log_2(x + 2)$

ACTIVITY 23 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 23-1

Let $g(x) = f^{-1}(x)$, the inverse of function f. Write the rule for g for each function f given below.

1. $f(x) = 7x - 9$
2. $f(x) = \left(\frac{1}{3}\right)^x$
3. $f(x) = 2x - 8$
4. $f(x) = -x + 3$
5. $f(x) = 5^x$
6. $f(x) = e^x$
7. $f(x) = \log_{20} x$
8. $f(x) = \ln x$

Simplify each expression.

9. $\log_3 3^x$
10. $12^{\log_{12} x}$
11. $\ln e^x$
12. $7^{\log_7 x}$

Lesson 23-2

Express each exponential statement as a logarithmic statement.

13. $12^2 = 144$
14. $2^{-3} = \frac{1}{8}$
15. $e^n = m$
16. $e^{3x} = 2$
17. $10^2 = 100$
18. $e^0 = 1$

Express each logarithmic statement as an exponential statement.

19. $\log_3 9 = 2$
20. $\log_2 64 = 6$
21. $\ln 1 = 0$
22. $\ln x = 6$
23. $\log_2 64 = 6$
24. $\ln e = 1$

Expand each expression. Assume all variables are positive.

25. $\log_2 x^2y^5$
26. $\log_4 \left(\frac{x^8}{5}\right)$
27. $\ln ex$
28. $\ln \left(\frac{1}{x}\right)$
29. Which is an equivalent form of the expression $\ln 5 + 2 \ln x$?
    A. $5 \ln x^2$
    B. $\ln 2x^5$
    C. $\ln 5x^2$
    D. $2 \ln x^5$

Rewrite each expression as a single, simplified logarithmic term. Assume all variables are positive.

30. $\log_2 32 + \log_2 2$

31. $\log_3 x^2 - \log_3 y$

32. $\ln x + \ln 2$

33. $3 \ln x$

Evaluate each expression without using a calculator.

34. $\log_{12} 12$

35. $\log_7 343$

36. $\log_7 49$

37. $\log_3 81$

Change each expression to a logarithmic expression in base 10. Use a calculator to find the value to three decimal places.

38. $\log_4 20$

39. $\log_{20} 4$

40. $\log_5 45$

41. $\log_3 18$

Lesson 23-3

42. If the domain of a logarithmic function is $(0, \infty)$ and the range is $(-\infty, \infty)$, what are the domain and range of the inverse of the function?
A. domain: $(-\infty, \infty)$, range: $(-\infty, \infty)$
B. domain: $(0, \infty)$, range: $(-\infty, \infty)$
C. domain: $(-\infty, \infty)$, range: $(-\infty, \infty)$
D. domain: $(-\infty, \infty)$, range: $(0, \infty)$

Graph each function, using a parent graph and the appropriate transformations. Describe the transformations.

43. $f(x) = 3 \log_2 (x) - 1$

44. $f(x) = \log_3 (x - 4) + 2$

45. $f(x) = \frac{1}{4} \log_4 x$

46. $f(x) = \log_2 (x + 3) - 4$

47. $f(x) = -2\log(x + 3) - 1$

48. $f(x) = -3\ln(x - 4) + 2$

MATHEMATICAL PRACTICES
Model with Mathematics

49. Given the function $f(x) = 2^x + 1$
a. Give the domain, range, y-intercept, and any asymptotes for $f(x)$. Explain.
b. Draw a sketch of the graph of the function on a grid. Describe the behavior of the function as x approaches ∞ and as x approaches $-\infty$.

College Costs

Lesson 24-1 Exponential Equations

My Notes

Learning Targets:

- Write exponential equations to represent situations.
- Solve exponential equations.

SUGGESTED LEARNING STRATEGIES: Summarizing, Paraphrasing, Create Representations, Vocabulary Organizer, Note Taking, Group Presentation

Wesley is researching college costs. He is considering two schools: a four-year private college where tuition and fees for the current year cost about $24,000, and a four-year public university where tuition and fees for the current year cost about $10,000. Wesley learned that over the last decade, tuition and fees have increased an average of 5.6% per year in four-year private colleges and an average of 7.1% per year in four-year public colleges.

To answer Items 1–4, assume that tuition and fees continue to increase at the same average rate per year as in the last decade.

1. Complete the table of values to show the estimated tuition for the next four years.

Years from Present	Private College Tuition and Fees	Public College Tuition and Fees
0	$24,000	$10,000
1		
2		
3		
4		

2. Express regularity in repeated reasoning. Write two functions to model the data in the table above. Let $R(t)$ represent the private tuition and fees and $U(t)$ represent the public tuition and fees, where t is the number of years from the present.

3. Wesley plans to be a senior in college six years from now. Use the models above to find the estimated tuition and fees at both the private and public colleges for his senior year in college.

4. Use appropriate tools strategically. Write an equation that can be solved to predict the number of years that it will take for the public college tuition and fees to reach the current private tuition and fees of $24,000. Find the solution using both the graphing and table features of a calculator.

MATH TIP

To solve an equation graphically on a calculator, enter each side of the equation as a separate function and find the intersection point of the two functions.

My Notes

Solving a problem like the one in Item 4 involves solving an exponential equation. An ***exponential equation*** is an equation in which the variable is in the exponent. Sometimes you can solve an exponential equation by writing both sides of the equation in terms of the same base. Then use the fact that when the bases are the same, the exponents must be equal:

$$b^m = b^n \text{ if and only if } m = n$$

Example A

Solve $6 \cdot 4^x = 96$.

$6 \cdot 4^x = 96$

Step 1: $4^x = 16$ — Divide both sides by 6.

Step 2: $4^x = 4^2$ — Write both sides in terms of base 4.

Step 3: $x = 2$ — If $b^m = b^n$, then $m = n$.

Example B

Solve $5^{4x} = 125^{x-1}$.

$5^{4x} = 125^{x-1}$

Step 1: $5^{4x} = (5^3)^{x-1}$ — Write both sides in terms of base 5.

Step 2: $5^{4x} = 5^{3x-3}$ — Power of a Power Property: $(a^m)^n = a^{mn}$

Step 3: $4x = 3x - 3$ — If $b^m = b^n$, then $m = n$.

Step 4: $x = -3$ — Solve for x.

Try These A–B

Solve for x. Show your work.

a. $3^x - 1 = 80$ **b.** $2^x = \frac{1}{32}$ **c.** $6^{3x-4} = 36^{x+1}$ **d.** $\left(\frac{1}{7}\right)^x = \left(\frac{1}{49}\right)$

MATH TIP

Check your work by substituting your solutions into the original problem and verifying the equation is true.

Check Your Understanding

5. When writing both sides of an equation in terms of the same base, how do you determine the base to use?

6. How could you check your solution to an exponential equation? Show how to check your answers to Try These part a.

LESSON 24-1 PRACTICE

Make use of structure. Solve for x by writing both sides of the equation in terms of the same base.

7. $2^{10x} = 32$

8. $4^x - 5 = 11$

9. $2^{4x-2} = 4^{x+2}$

10. $8^x = \frac{1}{64}$

11. $4 \cdot 5^x = 100$

12. $3 \cdot 2^x = 384$

13. $\left(\frac{1}{3}\right)^{2x} = \left(\frac{1}{9}\right)^{4-x}$

14. $\left(\frac{1}{2}\right)^{2x} = \left(\frac{1}{8}\right)^{10-x}$

15. Can you apply the method used in this lesson to solve the equation $2^{4x} = 27$? Explain why or why not.

My Notes

Learning Targets:

- Solve exponential equations using logarithms.
- Estimate the solution to an exponential equation.
- Apply the compounded interest formula.

SUGGESTED LEARNING STRATEGIES: Note Taking, Group Presentation, Create Representations, Close Reading, Vocabulary Organizer

For many exponential equations, it is not possible to rewrite the equation in terms of the same base. In this case, use the concept of inverses to solve the equation symbolically.

Example A

Estimate the solution of $3^x = 32$. Then solve to three decimal places.
Estimate that x is between 3 and 4, because $3^3 = 27$ and $3^4 = 81$.

$$3^x = 32$$

Step 1: $\log_3 3^x = \log_3 32$ — Take the log base 3 of both sides.

Step 2: $x = \log_3 32$ — Use the Inverse Property to simplify the left side.

Step 3: $x = \dfrac{\log 32}{\log 3}$ — Use the Change of Base Formula.

Step 4: $x \approx 3.155$ — Use a calculator to simplify.

Try These A

Estimate each solution. Then solve to three decimal places. Show your work.

a. $6^x = 12$ **b.** $5^x = 610$ **c.** $4^x = 0.28$ **d.** $e^x = 91$

MATH TIP

Recall that the *Inverse Properties* of logarithms state that for $b > 0, b \neq 1$:

$$\log_b b^x = x$$

and

$$b^{\log_b x} = x$$

Example B

Find the solution of $4^{x-2} = 35.6$ to three decimal places.

$$4^{x-2} = 35.6$$

$\log_4 4^{x-2} = \log_4 35.6$ — Take the log base 4 of both sides.

Step 1: $x - 2 = \log_4 35.6$ — Use the Inverse Property to simplify the left side.

Step 2: $x = \log_4 35.6 + 2$ — Solve for x.

Step 3: $x = \dfrac{\log 35.6}{\log 4} + 2$ — Use the Change of Base Formula.

Step 4: $x \approx 4.577$ — Use a calculator to simplify.

Try These B

Find each solution to three decimal places. Show your work.

a. $12^{x+3} = 240$ **b.** $4.2^{x+4} + 0.8 = 5.7$ **c.** $e^{2x-4} = 148$

My Notes

1. Rewrite the equation you wrote in Item 4 of Lesson 24-1. Then show how to solve the equation using the Inverse Property.

MATH TERMS

Compound interest is interest that is earned or paid not only on the principal but also on previously accumulated interest. At specific periods of time, such as daily or annually, the interest earned is added to the principal and then earns additional interest during the next period.

Wesley's grandfather gave him a birthday gift of $3000 to use for college. Wesley plans to deposit the money in a savings account. Most banks pay ***compound interest***, so he can use the formula below to find the amount of money in his savings account after a given period of time.

Compound Interest Formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

A = amount in account
P = principal invested
r = annual interest rate as a decimal
n = number of times per year that interest is compounded
t = number of years

MATH TIP

When interest is compounded annually, it is paid once a year. Other common compounding times are shown below.

Times per Year	
Semiannually	2
Quarterly	4
Monthly	12
Weekly	52
Daily	365

Example C

If Wesley deposits the gift from his grandfather into an account that pays 4% annual interest compounded quarterly, how much money will Wesley have in the account after three years?

Substitute into the compound interest formula. Use a calculator to simplify.

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = 3000\left(1 + \frac{0.04}{4}\right)^{4(3)} \approx \$3380.48$$

Solution: Wesley will have $3380.48 in the account after three years.

Try These C

How long would it take an investment of $5000 to earn $1000 interest if it is invested in a savings account that pays 3.75% annual interest compounded monthly?

Lesson 24-2
Solving Equations by Using Logarithms

My Notes

Wesley's grandfather recommends that Wesley deposit his gift into an account that earns interest compounded continuously, instead of at a fixed number of times per year.

Continuously Compounded Interest Formula

$A = Pe^{rt}$

A = amount in account
P = principal invested
r = annual interest rate as a decimal
t = number of years

Example D

If Wesley deposits the gift from his grandfather into an account that pays 4% annual interest compounded continuously, how much money will Wesley have in the account after three years?

Substitute into the continuously compounded interest formula. Use a calculator to simplify.

$$A = Pe^{rt} = 3000e^{0.04(3)} \approx \$3382.49$$

Solution: Wesley will have $3382.49 in the account after three years.

Try These D

How long would it take an investment of $5000 to earn $1000 interest if it is invested in a savings account that pays 3.75% annual interest compounded continuously?

Check Your Understanding

2. How is solving exponential and logarithmic equations similar to other equations that you have solved?

3. **Attend to precision.** In Examples C and D, why are the answers rounded to two decimal places?

4. A bank advertises an account that pays a monthly interest rate of 0.3% compounded continuously. What value do you use for r in the continuously compounded interest formula? Explain.

LESSON 24-2 PRACTICE

Solve for x to three decimal places.

5. $8^x = 100$

6. $3^{x-4} = 85$

7. $3e^{x+2} = 87$

8. $2^{3x-2} + 7 = 25$

9. $2 \cdot 4^{3x} - 3 = 27$

10. $e^{2x} - 1.5 = 6.7$

11. **Make sense of problems.** A deposit of $4000 is made into a savings account that pays 2.48% annual interest compounded quarterly.
   a. How much money will be in the account after three years?
   b. How long will it take for the account to earn $500 interest?
   c. How much more money will be in the account after three years if the interest is compounded continuously?

My Notes

Learning Targets:

- Solve logarithmic equations.
- Identify extraneous solutions to logarithmic equations.
- Use properties of logarithms to rewrite logarithmic expressions.

SUGGESTED LEARNING STRATEGIES: Create Representations, Vocabulary Organizer, Note Taking, Group Presentation

Equations that involve logarithms of variable expressions are called ***logarithmic equations***. You can solve some logarithmic equations symbolically by using the concept of functions and their inverses. Since the domain of logarithmic functions is restricted to the positive real numbers, it is necessary to check for ***extraneous solutions*** when solving logarithmic equations.

MATH TERMS

An **extraneous solution** is a solution that arises from a simplified form of the equation that does not make the original equation true.

Example A

Solve $\log_4 (3x - 1) = 2$.

$\log_4 (3x - 1) = 2$

Step 1: $4^{\log_4 (3x-1)} = 4^2$ — Write in exponential form using 4 as the base.

Step 2: $3x - 1 = 16$ — Use the Inverse Property to simplify the left side.

Step 3: $x = \frac{17}{3}$ — Solve for x.

Check: $\log_4 (3 \cdot \frac{17}{3} - 1) = \log_4 16 = 2$

Try These A

Solve for x. Show your work.

a. $\log_3 (x - 1) = 5$ **b.** $\log_2 (2x - 3) = 3$ **c.** $4 \ln (3x) = 8$

To solve other logarithmic equations, use the fact that when the bases are the same, $m > 0$, $n > 0$, and $b \neq 1$, the logarithmic values must be equal:

$$\log_b m = \log_b n \text{ if and only if } m = n$$

My Notes

Example B

Solve $\log_3 (2x - 3) = \log_3 (x + 4)$.

$\log_3 (2x - 3) = \log_3 (x + 4)$

Step 1: $2x - 3 = x + 4$ — If $\log_b m = \log_b n$, then $m = n$.

Step 2: $x = 7$ — Solve for x.

Check: $\log_3 (2 \cdot 7 - 3) \stackrel{?}{=} \log_3 (7 + 4)$

$\log_3 11 = \log_3 11$

Try These B

Solve for x. Check for extraneous solutions. Show your work.

a. $\log_6 (3x + 4) = 1$

b. $\log_5 (7x - 2) = \log_5 (3x + 6)$

c. $\ln 10 - \ln (4x - 6) = 0$

a. $3x + 4 = 6^1$
$3x = 2$
$x = \frac{2}{3}$

b. $7x - 2 = 3x + 6$
$4x = 8$
$x = 2$

c. $10 = 4x - 6$
$16 = 4x$
$x = 4$

Sometimes it is necessary to use properties of logarithms to simplify one side of a logarithmic equation before solving the equation.

Example C

Solve $\log_2 x + \log_2 (x + 2) = 3$.

$\log_2 x + \log_2 (x + 2) = 3$

Step 1: $\log_2 [x(x + 2)] = 3$ — Product Property of Logarithms

Step 2: $2^{\log_2 [x(x+2)]} = 2^3$ — Write in exponential form using 2 as the base.

Step 3: $x(x + 2) = 8$ — Use the Inverse Property to simplify.

Step 4: $x^2 + 2x - 8 = 0$ — Write as a quadratic equation.

Step 5: $(x + 4)(x - 2) = 0$ — Solve the quadratic equation.

Step 6: $x = -4$ or $x = 2$ — Check for extraneous solutions.

Check: $\log_2 (-4) + \log_2 (-4 + 2) \stackrel{?}{=} 3$

$\log_2 (-4) + \log_2 (-2) \stackrel{?}{=} 3$

$\log_2 2 + \log_2 (2 + 2) \stackrel{?}{=} 3$

$\log_2 2 + \log_2 4 \stackrel{?}{=} 3$

$\log_2 8 \stackrel{?}{=} 3$

$3 = 3$

Because $\log_2 (-4)$ and $\log (-2)$ are not defined, -4 is not a solution of the original equation; thus it is extraneous.

The solution is $x = 2$.

Try These C

Solve for x, rounding to three decimal places if necessary. Check for extraneous solutions.

a. $\log_4 (x + 6) - \log_4 x = 2$ $\frac{2}{5}$

b. $\ln (2x + 2) + \ln 5 = 2$ -0.261

c. $\log_2 2x + \log_2 (x - 3) = 3$ 4; -1 is extraneous

My Notes

Some logarithmic equations cannot be solved symbolically using the previous methods. A graphing calculator can be used to solve these equations.

Example D

Solve $-x = \log x$ using a graphing calculator.

$$-x = \log x$$

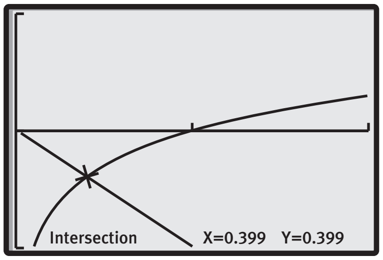
Step 1: Enter $-x$ for Y1.

Step 2: Enter $\log x$ for Y2.

Step 3: Graph both functions.

Step 4: Find the x-coordinate of the point of intersection: $x \approx 0.399$

Solution: $x \approx 0.399$



Try These D

Solve for x.

a. $x \log x = 3$ **b.** $\ln x = -x^2 - 1$ **c.** $\ln(2x + 4) = x^2$

Check Your Understanding

1. Explain how it is possible to have more than one solution to a simplified logarithmic equation, only one of which is valid.

2. Critique the reasoning of others. Than solves a logarithmic equation and gets two possible solutions, -2 and 4. Than immediately decides that -2 is an extraneous solution, because it is negative. Do you agree with his decision? Explain your reasoning.

LESSON 24-3 PRACTICE

Solve for x, rounding to three decimal places if necessary. Check for extraneous solutions.

3. $\log_5(3x + 4) = 2$

4. $\log_3(4x + 1) = 4$

5. $\log_{12}(4x - 2) = \log_{12}(x + 10)$

6. $\log_2 3 + \log_2(x - 4) = 4$

7. $\ln(x + 4) - \ln(x - 4) = 4$

8. Construct viable arguments. You saw in this lesson that logarithmic equations may have extraneous solutions. Do exponential equations ever have extraneous solutions? Justify your answer.

My Notes

Learning Targets:

- Solve exponential inequalities.
- Solve logarithmic inequalities.

SUGGESTED LEARNING STRATEGIES: Note Taking, Group Presentation, Create Representations

You can use a graphing calculator to solve exponential and logarithmic inequalities.

Example A

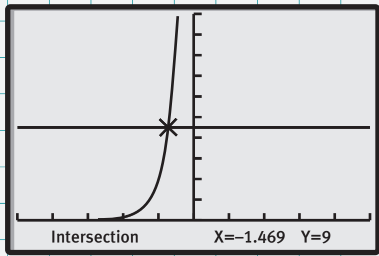
Use a graphing calculator to solve the inequality $4.2^{x+3} > 9$.

Step 1: Enter 4.2^{x+3} for Y1 and 9 for Y2.

Step 2: Find the x-coordinate of the point of intersection: $x \approx -1.469$

Step 3: The graph of $y = 4.2^{x+3}$ is above the graph of $y = 9$ when $x > -1.469$.

Solution: $x > -1.469$



Try These A

Use a graphing calculator to solve each inequality.

a. $3 \cdot 5.1^{1-x} < 75$ **b.** $\log 10x \geq 1.5$ **c.** $7.2 \ln x + 3.9 \leq 12$

Example B

Scientists have found a relationship between atmospheric pressure and altitudes up to 50 miles above sea level that can be modeled by $P = 14.7(0.5)^{\frac{a}{3.6}}$. P is the atmospheric pressure in lb/in.^{2} Solve the equation $P = 14.7(0.5)^{\frac{a}{3.6}}$ for a. Use this equation to find the atmospheric pressure when the altitude is greater than 2 miles.

Step 1: Solve the equation for a.

$$\frac{P}{14.7} = 0.5^{\frac{a}{3.6}}$$ Divide both sides by 14.7.

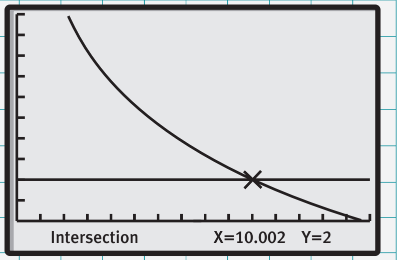
$$\log_{0.5}\left(\frac{P}{14.7}\right) = \log_{0.5}\left(0.5^{\frac{a}{3.6}}\right)$$ Take the log base 0.5 of each side.

$$\log_{0.5}\left(\frac{P}{14.7}\right) = \frac{a}{3.6}$$ Simplify.

$$3.6\log_{0.5}\left(\frac{P}{14.7}\right) = a$$ Multiply both sides by 3.6.

$$\frac{3.6\log\left(\frac{P}{14.7}\right)}{\log 0.5} = a$$ Use the Change of Base Formula.

My Notes



Step 2: Use your graphing calculator to solve the inequality

$$\frac{3.6\log\left(\frac{P}{14.7}\right)}{\log 0.5} > 2.$$

The graph of $y = \frac{3.6\log\left(\frac{x}{14.7}\right)}{\log 0.5}$ is above the graph of $y = 2$ when $0 < x < 10.002$.

Solution: When the altitude is greater than 2, the atmospheric pressure is between 0 and 10.002 lb/in.^{2}.

Try These B

Suppose that the relationship between C, the number of digital cameras supplied, and the price x per camera in dollars is modeled by the function $C = -400 + 180 \cdot \log x$.

a. Find the range in the price predicted by the model if there are between 20 and 30 cameras supplied.

b. Solve the equation for x. Use this equation to find the number of cameras supplied when the price per camera is more than \$300.

Check Your Understanding

2. How are exponential and logarithmic inequalities different from exponential and logarithmic equations?

3. Describe how to find the solution of an exponential or logarithmic inequality from a graph. What is the importance of the intersection point in this process?

LESSON 24-4 PRACTICE

Use a graphing calculator to solve each inequality.

4. $16.4(0.87)^{x-1.5} \geq 10$

5. $30 < 25\log(3.5x - 4) + 12.6 < 50$

6. $4.5e^x \leq 2$

7. $\ln(x - 7.2) > 1.35$

ACTIVITY 24 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 24-1

1. Which exponential equation can be solved by rewriting both sides in terms of the same base?
   A. $4^x = 12$
   B. $6 \cdot 2^{x-3} = 256$
   C. $3^{x+2} - 5 = 22$
   D. $e^x = 58$

2. Solve for x.
   a. $16^x = 32^{x-1}$
   b. $8 \cdot 3^x = 216$
   c. $5^x = \frac{1}{625}$
   d. $7^{2x} = 343^{x-4}$
   e. $4^x + 8 = 72$
   f. $e^x = 3$
   g. $e^{3x} = 2$
   h. $3e^{5x} = 42$

Lesson 24-2

3. Solve for x to three decimal places.
   a. $7^x = 300$
   b. $5^{x-4} = 135$
   c. $3^{2x+1} - 5 = 80$
   d. $3 \cdot 6^{3x} = 0.01$
   e. $5^x = 212$
   f. $3(2^{x+4}) = 350$

4. A deposit of $1000 is made into a savings account that pays 4% annual interest compounded monthly.
   a. How much money will be in the account after 6 years?
   b. How long will it take for the $1000 to double?

5. June invests $7500 at 12% interest for one year.
   a. How much would she have if the interest is compounded yearly?
   b. How much would she have if the interest is compounded daily?

6. If $4000 is invested at 7% interest per year compounded continuously, how long will it take to double the original investment?

7. At what annual interest rate, compounded continuously, will money triple in nine years?
   A. 1.3%
   B. 7.3%
   C. 8.1%
   D. 12.2%

Lesson 24-3

8. Compare the methods of solving equations in the form of *log* = *log* (such as $\log_3 (2x - 3) = \log_3 (x + 4)$) and *log* = *number* (such as $\log_4 (3x - 1) = 2$).

9. Solve for x. Check for extraneous solutions.
   a. $\log_2 (5x - 2) = 3$
   b. $\log_4 (2x - 3) = 2$
   c. $\log_7 (5x + 3) = \log_7 (3x + 11)$
   d. $\log_6 4 + \log_6 (x + 2) = 1$
   e. $\log_3 (x + 8) = 2 - \log_3 (x)$
   f. $\log_2 (x + 6) - \log_2(x) = 3$
   g. $\log_2 x - \log_2 5 = \log_2 10$
   h. $5 \ln 3x = 40$
   i. $\ln 4x = 30$

10. If an equation contains
  a. $\log(x - 2)$, how do you know the solutions must be greater than 2?
  b. $\log(x + 3)$, how do you know solutions must be greater than -3?

11. Solve for x to three decimal places using a graphing calculator.
  a. $\ln 3x = x^2 - 2$
  b. $\log(x + 7) = x^2 - 6x + 5$

Lesson 24-4

12. Use a graphing calculator to solve each inequality.
  a. $2000 < 1500(1.04)^{12x} < 3000$
  b. $4.5 \log(2x) + 8.4 \geq 9.2$
  c. $\log_3(3x - 5) \geq \log_3(x + 7)$
  d. $\log_2 2x \leq \log_4(x + 3)$
  e. $5^{x+3} \leq 2^{x+4}$

MATHEMATICAL PRACTICES
Look For and Make Use of Structure

13. Explore how the compounded interest formula is related to the continuously compounded interest formula.
  a. Consider the expression $\left(1 + \frac{1}{m}\right)^m$, where m is a positive integer. Enter the expression in your calculator as y_1. Then find the value of $y_1(1000)$, $y_1(10{,}000)$, and $y_1(1{,}000{,}000)$.
  b. As m increases, what happens to the value of the expression?
  c. The compounded interest formula is $A = P\left(1 + \frac{r}{n}\right)^{nt}$. Let $m = \frac{n}{r}$. Explain why the formula may be written as

$$A = P\left[\left(1 + \frac{1}{m}\right)^m\right]^{rt}.$$

  d. As the number of compounding periods, n, increases, so does the value of m. Explain how your results from parts b and c show the connection between the compounded interest formula and the continuously compounded interest formula.

Exponential and Logarithmic Equations
EVALUATING YOUR INTEREST

1. **Make use of structure.** Express each exponential statement as a logarithmic statement.
   a. $5^{-3} = \frac{1}{125}$
   b. $7^2 = 49$
   c. $20^2 = 400$
   d. $3^6 = 729$
2. Express each logarithmic statement as an exponential statement.
   a. $\log_8 512 = 3$
   b. $\log_9\left(\frac{1}{729}\right) = -3$
   c. $\log_2 64 = 6$
   d. $\log_{11} 14{,}641 = 4$
3. Evaluate each expression without using a calculator.
   a. $25^{\log_{25} x}$
   b. $\log_3 3^x$
   c. $\log_3 27$
   d. $\log_8 1$
   e. $\log_2 40 - \log_2 5$
   f. $\frac{\log 25}{\log 5}$
4. Solve each equation symbolically. Give approximate answers rounded to three decimal places. Check your solutions. Show your work.
   a. $4^{2x-1} = 64$
   b. $5^x = 38$
   c. $3^{x+2} = 98.7$
   d. $2^{3x-4} + 7.5 = 23.6$
   e. $\log_3(2x + 1) = 4$
   f. $\log_8(3x - 2) = \log_8(x + 1)$
   g. $\log_2(3x - 2) + \log_2 8 = 5$
   h. $\log_6(x - 5) + \log_6 x = 2$
5. Let $f(x) = \log_2(x - 1) + 3$.
   a. Sketch a parent graph and a series of transformations that result in the graph of f. How would the graph of $y = \log(x - 1) + 3$ and $y = \ln(x - 1) + 3$ compare?
   b. Give the equation of the vertical asymptote of the graph of f.
6. **Make sense of problems.** Katie deposits $10,000 in a savings account that pays 8.5% interest per year, compounded quarterly. She does not deposit more money and does not withdraw any money.
   a. Write the formula to find the amount in the account after 3 years.
   b. Find the total amount she will have in the account after 3 years.
7. How long would it take an investment of $6500 to earn $1200 interest if it is invested in a savings account that pays 4% annual interest compounded quarterly? Show the solution both graphically and symbolically.

Exponential and Logarithmic Equations

EVALUATING YOUR INTEREST

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
	The solution demonstrates these characteristics:			
Mathematics Knowledge and Thinking (Items 1–7)	• Fluency and accuracy in evaluating and rewriting exponential and logarithmic equations and expressions • Effective understanding of and accuracy in solving logarithmic and exponential equations algebraically and graphically • Effective understanding of logarithmic functions and their key features as transformations of a parent graph	• Largely correct work when evaluating and rewriting exponential and logarithmic equations and expressions • Adequate understanding of how to solve logarithmic and exponential equations algebraically and graphically leading to solutions that are usually correct • Adequate understanding of logarithmic functions and their key features as transformations of a parent graph	• Difficulty when evaluating and rewriting logarithmic and exponential equations and expressions • Partial understanding of how to solve logarithmic and exponential equations algebraically and graphically • Partial understanding of logarithmic functions and their key features as transformations of a parent graph	• Mostly inaccurate or incomplete work when evaluating and rewriting logarithmic and exponential equations and expressions • Inaccurate or incomplete understanding of how to solve exponential and logarithmic equations algebraically and graphically • Little or no understanding of logarithmic functions and their key features as transformations of a parent graph
Problem Solving (Items 6, 7)	• An appropriate and efficient strategy that results in a correct answer	• A strategy that may include unnecessary steps but results in a correct answer	• A strategy that results in some incorrect answers	• No clear strategy when solving problems
Mathematical Modeling / Representations (Items 5–7)	• Fluency in modeling a real-world scenario with an exponential equation or graph • Effective understanding of how to graph a logarithmic function using transformations	• Little difficulty in accurately modeling a real-world scenario with an exponential equation or graph • Largely correct understanding of how to graph a logarithmic function using transformations	• Some difficulty in modeling a real-world scenario with an exponential equation or graph • Partial understanding of how to graph a logarithmic function using transformations	• Significant difficulty with modeling a real-world scenario with an exponential equation or graph • Mostly inaccurate or incomplete understanding of how to graph a logarithmic function using transformations
Reasoning and Communication (Items 6, 7)	• Clear and accurate use of mathematical work to justify an answer	• Correct use of mathematical work to justify an answer	• Partially correct justification of an answer using mathematical work	• Incorrect or incomplete justification of an answer using mathematical work