Polynomials

Unit Overview

In this unit you will study polynomials, beginning with real-world applications and polynomial operations. You will also investigate intercepts, end behavior, and relative extrema. You will learn to apply the Binomial Theorem to expand binomials, and you will be introduced to several theorems that will assist you in factoring, graphing, and understanding polynomial functions.

Key Terms

As you study this unit, add these and other terms to your math notebook. Include in your notes your prior knowledge of each word, as well as your experiences in using the word in different mathematical examples. If needed, ask for help in pronouncing new words and add information on pronunciation to your math notebook. It is important that you learn new terms and use them correctly in your class discussions and in your problem solutions.

Academic Vocabulary

- alternative

Math Terms

- polynomial function
- degree
- standard form of a polynomial
- relative maximum
- relative minimum
- end behavior
- even function
- odd function
- synthetic division
- combination
- factorial
- summation notation
- Fundamental Theorem of Algebra
- extrema
- relative extrema
- global extrema

EMBEDDED ASSESSMENTS

This unit has two embedded assessments, following Activities 16 and 18. The first will give you the opportunity to demonstrate what you have learned about polynomial functions, including operations on polynomials. You will also be asked to apply the Binomial Theorem. The second assessment focuses on factoring and graphing polynomial functions.

**Write your answers on notebook paper.
Show your work.**

1. Find the surface area and volume of a rectangular prism formed by the net below. The length is 10 units, the width is 4 units, and the height is 5 units.
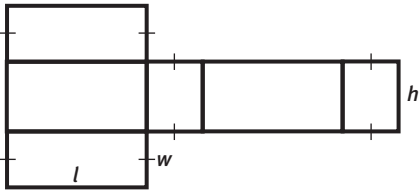
2. Simplify $(2x^2 + 3x + 7) - (4x - 2x^2 + 9)$.
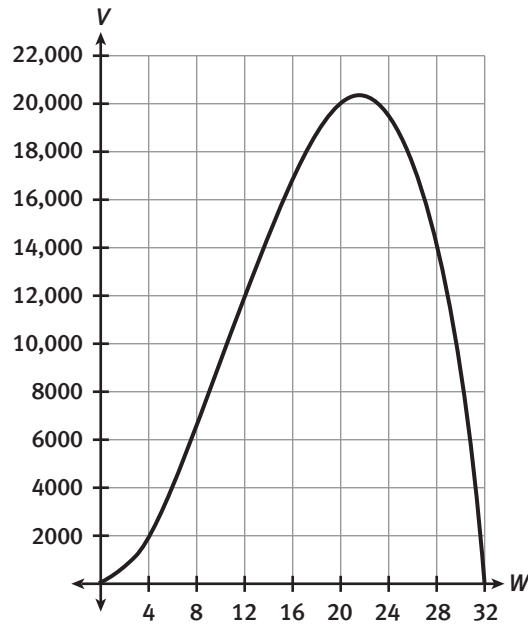
3. Factor $9x^4 - 49x^2y^2$.

4. Factor $2x^2 - 9x - 5$.

5. Simplify $(x + 4)^4$.

6. Given a function $f(x) = 3x^4 - 5x^2 + 2x - 3$, evaluate $f(-1)$.

7. Find the *x*- and *y*-intercepts of the graph whose equation is $y = 3x - 12$.

8. Determine whether the graph below is symmetric. If it is, describe the symmetry.
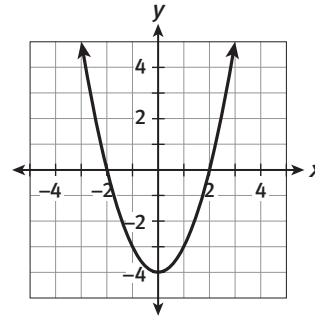
9. The graph below represents $f(x)$. Find $f(28)$.

Introduction to Polynomials

Postal Service
Lesson 14-1 Polynomials

My Notes

Learning Targets:
- Write a third-degree equation that represents a real-world situation.
- Graph a portion of this equation and evaluate the meaning of a relative maximum.

> **SUGGESTED LEARNING STRATEGIES:** Create Representations, Note Taking, Think-Pair-Share

The United States Postal Service will not accept rectangular packages if the perimeter of one end of the package plus the length of the package is greater than 130 in. Consider a rectangular package with square ends as shown in the figure.
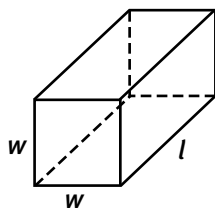
1. Work with your group on this item and on Items 2–5. Assume that the perimeter of one end of the package plus the length of the package equals the maximum 130 in. Complete the table with some possible measurements for the length and width of the package. Then find the corresponding volume of each package.

Width (in.)	Length (in.)	Volume (in.^{3})

2. Give an estimate for the largest possible volume of an acceptable United States Postal Service rectangular package with square ends.

DISCUSSION GROUP TIP

Reread the problem scenario as needed. Make notes on the information provided in the problem. Respond to questions about the meaning of key information. Summarize or organize the information needed to create reasonable solutions, and describe the mathematical concepts your group will use to create its solutions.

My Notes

3. **Model with mathematics.** Use the package described in Item 1.
   a. Write an expression for ℓ, the length of the package, in terms of w, the width of the square ends of the package.

   b. Write the volume of the package V as a function of w, the width of the square ends of the package.

   c. Justify your answer by explaining what each part of your equation represents. As you justify your answer, speak clearly and use precise mathematical language to describe your reasoning and your conclusions. Remember to use complete sentences, including transitions and words such as *and, or, since, for example, therefore, because of* to make connections between your thoughts.

4. Consider the smallest and largest possible values for w that make sense for the function you wrote in Item 3b. Give the domain of the function as a model of the volume of the postal package. Express the domain as an inequality, in interval notation, and in set notation.

5. Sketch a graph of the function in Item 3b over the domain that you found in Item 4. Include the scale on each axis.
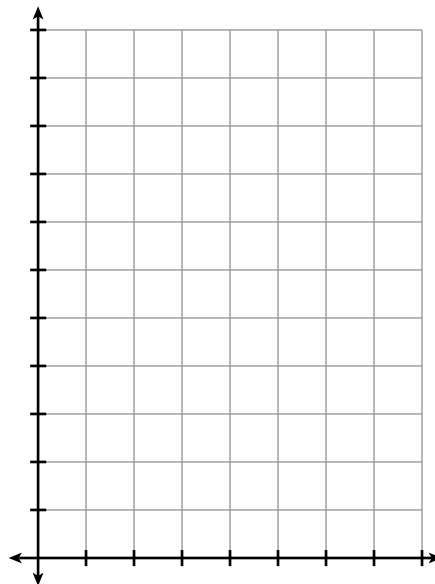
6. **Use appropriate tools strategically.** Use a graphing calculator to find the coordinates of the maximum point of the function that you graphed in Item 5.

7. What information do the coordinates of the maximum point of the function found in Item 6 provide with respect to an acceptable United States Postal Service package with square ends?

CONNECT **TO** **TECHNOLOGY**

Graphing calculators will allow you to find the maximum and minimum of functions in the graphing window.

CONNECT **TO** **AP**

In calculus, you will learn about the derivative of a function, which can be used to find the maximum and minimum values of a function.

Check Your Understanding

8. Explain why the function $V(w)$ that you used in this lesson is a third-degree equation.

9. Explain why the value of w cannot equal 0 in this situation.

10. Explain why the value of w must be strictly less than 32.5 in this situation.

11. In this situation, is it possible for the range of the function to be all real numbers? Why or why not?

12. **Critique the reasoning of others.** Another method of shipping at the Post Office allows for the perimeter of one end of a box plus the length of that box to be no greater than 165 inches. Sarena wants to ship a box whose height is twice the width using this method. She says the formula for the volume of such a box is $V(w) = (165 - 6w)2w^2$. Her sister, Monique, says the formula is $V(w) = (165 - w)w^2$. Who is right? Justify your response.

My Notes

LESSON 14-1 PRACTICE

13. The volume of a rectangular box is given by the function $V(w) = (60 - 4w)w^2$. What is a reasonable domain for the function in this situation? Express the domain as an inequality, in interval notation, and in set notation.

14. Sketch a graph of the function in Item 13 over the domain that you found. Include the scale on each axis.

15. Use a graphing calculator to find the coordinates of the maximum point of the function given in Item 13.

16. What is the width of the box, in inches, that produces the maximum volume?

17. **Reason abstractly.** An architect uses a cylindrical tube to ship blueprints to a client. The height of the tube plus twice its radius must be less than 60 cm.
    a. Write an expression for h, the height of the tube, in terms of r, the radius of the tube.
    b. Write an expression for V, the volume of the tube, in terms of r, the radius of the tube.
    c. Find the radius that produces the maximum volume.
    d. Find the maximum volume of the tube.

Learning Targets:
- Sketch the graphs of cubic functions.
- Identify the end behavior of polynomial functions.

> **SUGGESTED LEARNING STRATEGIES:** Vocabulary Organizer, Marking the Text, Create Representations, Predict and Confirm

When using a function to model a given situation, such as the acceptable United States Postal Service package, the reasonable domain may be only a portion of the domain of real numbers. Moving beyond the specific situation, you can examine the *polynomial function* across the domain of the real numbers.

A **polynomial function** in one variable is a function that can be written in the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where n is a nonnegative integer, the coefficients $a_0, a_1, \ldots a_n$ are real numbers, and $a_n \neq 0$. The highest power, n, is the **degree** of the polynomial function.

A polynomial is in **standard form** when all like terms have been combined, and the terms are written in descending order by exponent.

Leading coefficient   Degree of the polynomial

$$7x^5 + 2x^2 - 3$$
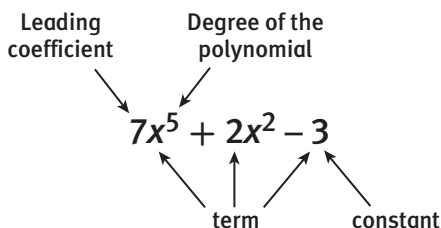
term   constant

Various attributes of the graph of a polynomial can be predicted by its equation. Here are some examples:

- the constant term is the *y*-intercept of the graph;
- the degree tells the maximum number of *x*-intercepts the graph of a polynomial can have; and
- the degree of the polynomial gives you information about the shape of the graph at its ends.
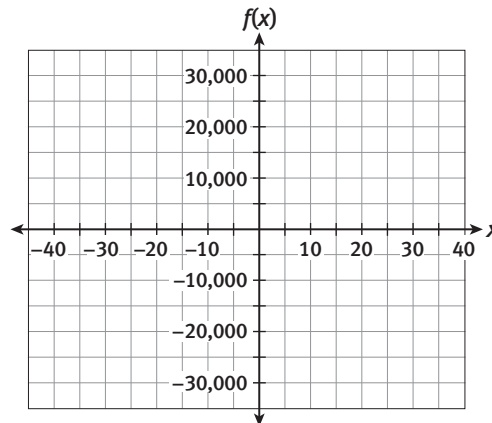
My Notes

MATH TERMS

Some common types of **polynomial functions** are listed in the table. You are already familiar with some of these.

Polynomial functions are named by the **degree** of the function.

Degree	Name
0	Constant
1	Linear
2	Quadratic
3	Cubic
4	Quartic

My Notes

1. Write a polynomial function $f(x)$ defined over the set of real numbers in standard form such that it has the same function rule as $V(w)$, the rule you found in Item 3b of the previous lesson for the volume of the rectangular box. Sketch a graph of the function.

$f(x)$

(graph with vertical axis labeled 30,000; 20,000; 10,000; −10,000; −20,000; −30,000 and horizontal axis x labeled −40, −30, −20, −10, 10, 20, 30, 40)

MATH TERMS

A function value $f(a)$ is called a **relative maximum** of f if there is an interval around a where, for any x in the interval, $f(a) \geq f(x)$.

A function value $f(a)$ is called a **relative minimum** of f if there is an interval around a where, for any x in the interval, $f(a) \leq f(x)$.

2. Name any *relative maximum* values and *relative minimum* values of the function $f(x)$ in Item 1.

3. Name any x- or y-intercepts of the function $f(x) = -4x^3 + 130x^2$.

MATH TIP

Think of a *relative minimum* of a graph as being the bottom of a hill and the *relative maximum* as the top of a hill.

4. **Model with mathematics.** Use a graphing calculator to sketch a graph of the cubic function $f(x) = 2x^3 - 5x^2 - 4x + 12$.

$f(x)$

(graph with vertical axis labeled 15, 10, 5, −5, −10, −15 and horizontal axis x labeled −10, −8, −6, −4, −2, 2, 4, 6, 8, 10)

TECHNOLOGY TIP

When the coefficients of an equation are relatively small, begin with a standard 10-by-10 viewing window, and then adjust the window if necessary.

5. Name any relative maximum values and relative minimum values of the function $f(x)$ in Item 4.

6. Name any x- or y-intercepts of the function in Item 4.

Check Your Understanding

7. Decide if the function $f(x) = 7x - 2 + x^2 - 4x^3$ is a polynomial. If it is, write the function in standard form and then state the degree and leading coefficient.

8. **Construct viable arguments.** Explain why $f(x) = 2x + 5 - \dfrac{1}{x}$ is not a polynomial.

9. Use a graphing calculator to sketch a graph of the cubic function $f(x) = x^3 + x^2 - 4x - 2$.

10. Use a graphing calculator to determine how many x-intercepts the graph of $f(x) = x^3 + x^2 - 4x + 5$ has.

11. **Use appropriate tools strategically.** Use the graphs you have sketched in this lesson to speculate about the minimum number of times a cubic function must cross the x-axis and the maximum number of times it can cross the x-axis.

The ***end behavior*** of a graph is the appearance of the graph on the extreme right and left ends of the x-axis. That is, you look to see what happens to y as x approaches $-\infty$ and ∞.

Examine your graph from Item 1. To describe the end behavior of the graph, you would say: The left side of the graph increases (points upward) continuously and the right side of the graph decreases (points downward) continuously. You can also use mathematical notation, called *arrow notation*, to describe end behavior. For this graph you would write:
As $x \to -\infty$, $y \to \infty$, and as $x \to \infty$, $y \to -\infty$.

12. Examine your graph from Item 4. Describe the end behavior of the graph in words and by using arrow notation.

MATH TERMS

End behavior describes what happens to a graph at the extreme ends of the x-axis, as x approaches $-\infty$ and ∞.

MATH TIP

Recall that the phrase *approaches positive infinity* or *approaches ∞* means "increases continuously," and that approaches *negative infinity* or *approaches $-\infty$* means "decreases continuously."

Values that increase or decrease continuously, or without stopping, are said to increase or decrease *without bound*.

My Notes
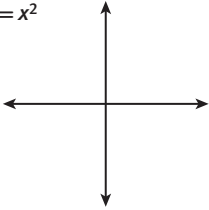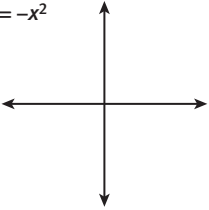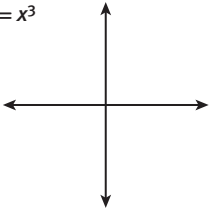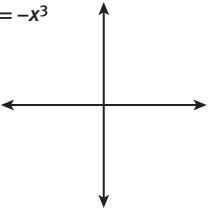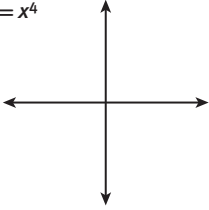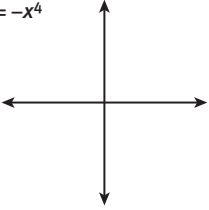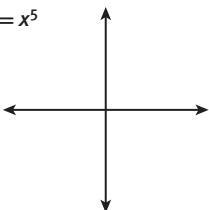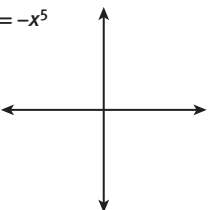
13. Examine the end behavior of $f(x) = 3x^2 - 6$.
   a. As x goes to ∞, what behavior does the function have?

   b. How is the function behaving as x approaches $-\infty$?

It is possible to determine the end behavior of a polynomial's graph simply by looking at the degree of the polynomial and the sign of the leading coefficient.

MATH TIP

The leading term of a polynomial (which has the greatest power when the polynomial is written in standard form) determines the end behavior. Learning these basic polynomial shapes will help you describe the end behavior of any polynomial.

14. Use appropriate tools strategically. Use a graphing calculator to examine the *end behavior* of polynomial functions in general. Sketch each given function on the axes below.

a. $y = x^2$	**b.** $y = -x^2$
c. $y = x^3$	**d.** $y = -x^3$
e. $y = x^4$	**f.** $y = -x^4$
g. $y = x^5$	**h.** $y = -x^5$

Some Attributes of Polynomial Functions

My Notes

15. Which of the functions in Item 14 have the same end behavior on the right side of the graph as on the left side?

16. Reason quantitatively. What is true about the degree of each of the functions you identified in Item 15?

17. Make a conjecture about how the degree affects the end behavior of polynomial functions.

18. For which of the functions that you identified in Item 15 does the end behavior decrease without bound on both sides of the graph?

19. What is true about the leading coefficient of each of the functions you identified in Item 18?

20. Express regularity in repeated reasoning. Work with your group. Make a conjecture about how the sign of the leading coefficient affects the end behavior of polynomial functions.

DISCUSSION GROUP TIPS

Read the text carefully to clarify meaning. Reread definitions of terms as needed, or ask your teacher to clarify vocabulary terms.

If you need help in describing your ideas during group discussions, make notes about what you want to say. Listen carefully to other group members and ask for clarification of meaning for any words routinely used by group members.

My Notes

Check Your Understanding

21. Use arrow notation to describe the left-end behavior of a graph that decreases without bound.

22. Describe in words the end behavior of a graph that is described by the following arrow notation: As $x \to \pm\infty$, $y \to -\infty$.

23. Reason abstractly. If the end behavior of a graph meets the description in Item 22, is it possible that the graph represents a third-degree polynomial? Explain your answer.

24. Give two examples of a polynomial whose graph increases without bound as x approaches both positive and negative infinity.

LESSON 14-2 PRACTICE

25. Sketch the graph of the polynomial function $f(x) = x^3 - 6x^2 + 9x$.

26. Name any x-intercepts, y-intercepts, relative maximums, and relative minimums for the function in Item 25.

27. Make sense of problems. Sketch a graph of any third-degree polynomial function that has three distinct x-intercepts, a relative minimum at $(-6, -4)$, and a relative maximum at $(3, 5)$.

28. Decide if each function is a polynomial. If it is, write the function in standard form, and then state the degree and leading coefficient.
  a. $f(x) = 5x - x^3 + 3x^5 - 2$
  b. $f(x) = -\dfrac{2}{3}x^3 - 8x^4 - 2x + 7$
  c. $f(x) = 4^x + 2x^2 + x + 5$

29. Describe the end behavior of each function.
  a. $f(x) = x^6 - 2x^3 + 3x^2 + 2$
  b. $f(x) = -\dfrac{2}{3}x^3 - 8x^2 - 2x + 7$

Learning Targets:

- Recognize even and odd functions given an equation or graph.
- Distinguish between even and odd functions and even-degree and odd-degree functions.

> **SUGGESTED LEARNING STRATEGIES:** Paraphrasing, Marking the Text, Create Representations

The graphs of some polynomial functions have special attributes that are determined by the value of the exponents in the polynomial.

1. Graph the functions $f(x) = 3x^2 + 1$ and $f(x) = 2x^3 + 3x$ on the axes.

> **MATH TIP**
>
> The graph of a function can be symmetric across an axis or other line when the graph forms a mirror image across the line. The graph can be symmetric around a point when rotation of the graph can superimpose the image on the original graph.
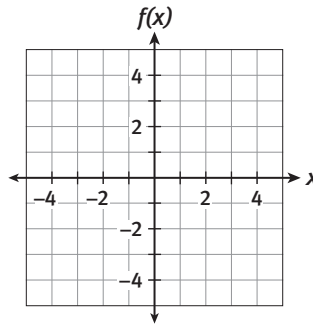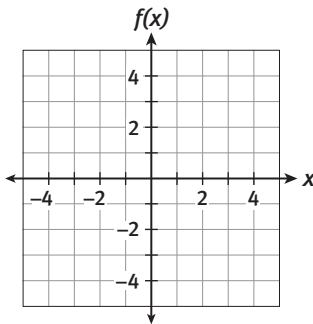
2. Describe the symmetry of the graph of $f(x) = 3x^2 + 1$.

3. Describe the symmetry of the graph of $f(x) = 2x^3 + 3x$.

The function $f(x) = 3x^2 + 1$ is called an **even function**. Notice that *every* power of x is an even number—there is no x^1 term. This is true for the constant term as well, since you can write a constant term as the coefficient of x^0. Symmetry over the y-axis is an attribute of all even functions.

The function $f(x) = 2x^3 + 3x$ is an **odd function**. Notice that every power of x is an odd number—there is no x^2 or constant (x^0) term. Symmetry around the origin is an attribute of all odd functions.

> **MATH TERMS**
>
> Algebraically, an **even function** is one in which $f(-x) = f(x)$.
>
> An **odd function** is one in which $f(-x) = -f(x)$.

My Notes

4. Examine the sketches you made in Item 14 of the previous lesson. Use symmetry to determine which graphs are even functions and which are odd functions. Explain your reasoning.

5. **Make use of structure.** Explain how an examination of the equations in Item 14 of the previous lesson supports your answer to Item 4.

Check Your Understanding

6. Explain why the function $f(x) = 4x^2 + 8x$ is neither even nor odd.

7. For a given polynomial function, as x approaches $-\infty$ the graph increases without bound, and as x approaches ∞ the graph decreases without bound. Is it possible that this function is an even function? Explain your reasoning.

LESSON 14-3 PRACTICE

8. Determine whether the function $f(x) = 2x^5 + 3x^3 + 7$ is even, odd, or neither. Explain your reasoning.

9. Determine whether the function below is even, odd, or neither. Justify your answer.
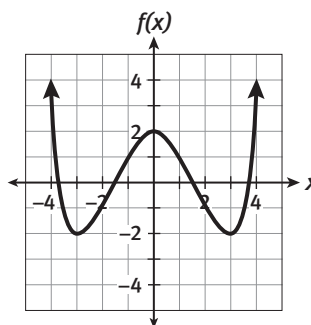
10. **Attend to precision.** Give an example of a polynomial function that has an odd degree, but is not an odd function.

ACTIVITY 14 PRACTICE

Write your answers on notebook paper.
Show your work.

Lesson 14-1

1. The volume of a rectangular box is given by the expression $V = (120 - 6w)w^2$, where w is measured in inches.
   a. What is a reasonable domain for the function in this situation? Express the domain as an inequality, in interval notation, and in set notation.
   b. Sketch a graph of the function over the domain that you found. Include the scale on each axis.
   c. Use a graphing calculator to find the coordinates of the maximum point of the function.
   d. What is the width of the box, in inches, that produces the maximum volume?

2. A cylindrical can is being designed for a new product. The height of the can plus twice its radius must be 45 cm.
   a. Find an equation that represents the volume of the can, given the radius.
   b. Find the radius that yields the maximum volume.
   c. Find the maximum volume of the can.

Lesson 14-2

3. Sketch the graph of the polynomial function $f(x) = -x^3 + 4x^2 - 4x$.

4. Name any x- or y-intercepts of the function $f(x)$ in Item 3.

5. Name any relative maximum values and relative minimum values of the function $f(x)$ in Item 3.

For Items 6–10, decide if each function is a polynomial. If it is, write the function in standard form, and then state the degree and leading coefficient.

6. $f(x) = 7x^2 - 9x^3 + 3x^7 - 2$

7. $f(x) = 2x^3 + x - 5^x + 9$

8. $f(x) = x^4 + x + 5 - \dfrac{1}{4}x^3$

9. $f(x) = -0.32x^3 + 0.08x^4 + 5^{x-1} - 3$

10. $f(x) = 3x + 5 + \sqrt{x}$

11. Examine the graph below.
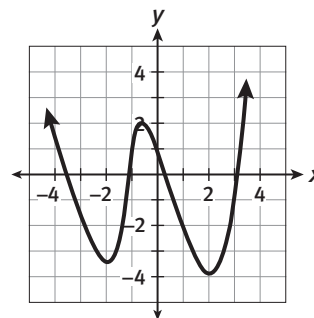
Which of the following statements is NOT true regarding the polynomial whose graph is shown?

A. The degree of the polynomial is even.
B. The leading coefficient is positive.
C. The function is a second-degree polynomial.
D. As $x \to \pm\infty$, $y \to \infty$.

For Items 12 and 13, describe the end behavior of each function using arrow notation.

12. $f(x) = x^6 - 2x^3 + 3x^2 + 2$

13. $f(x) = -x^3 + 7x^2 - 11$

14. Use the concept of end behavior to explain why a third-degree polynomial function must have at least one x-intercept.

15. Sketch a graph of any third-degree polynomial function that has exactly one x-intercept, a relative minimum at $(-2, 1)$, and a relative maximum at $(4, 3)$.

Lesson 14-3

For Items 16–28, determine whether each function is even, odd, or neither.

16. $f(x) = 10 + 3x^2$

17. $f(x) = -x^3 + 2x + 5$

18. $f(x) = 6x^5 - 4x$

19. When graphed, which of the following polynomial functions is symmetric about the origin?
 A. $f(x) = -x^3 + 2x + 5$
 B. $f(x) = x^3 + 8x$
 C. $f(x) = -7x^2 + 5$
 D. $f(x) = 5x^3 + 3x^2 - 7x + 1$

20. Sketch a graph of an even function whose degree is greater than 2.

21. If $f(x)$ is an even function and passes through the point (5, 3), what other point must lie on the graph of the function? Explain your reasoning.

MATHEMATICAL PRACTICES
Construct Viable Arguments and Critique the Reasoning of Others

22. Sharon described the function graphed below as follows:
 • It is a polynomial function.
 • It is an even function.
 • It has a positive leading coefficient.
 • The degree n could be any even number greater than or equal to 2.

Critique Sharon's description. If you disagree with any of her statements, provide specific reasons as to why.

Polynomial Operations
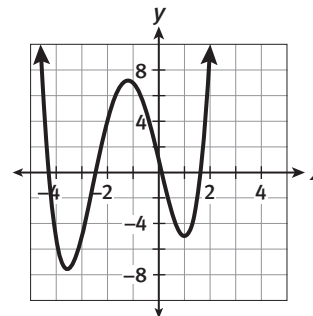
Polly's Pasta
Lesson 15-1 Adding and Subtracting Polynomials

Learning Targets:
- Use a real-world scenario to introduce polynomial addition and subtraction.
- Add and subtract polynomials.

> **SUGGESTED LEARNING STRATEGIES:** Create Representations, Think-Pair-Share, Discussion Groups, Self Revision/Peer Revision

Polly's Pasta and Pizza Supply sells wholesale goods to local restaurants. They keep track of revenue earned from selling kitchen supplies and food products. The function K models revenue from kitchen supplies and the function F models revenue from food product sales for one year in dollars, where t represents the number of the month (1–12) on the last day of the month.

$$K(t) = 15t^3 - 312t^2 + 1600t + 1100$$
$$F(t) = 36t^3 - 720t^2 + 3800t - 1600$$

1. What kind of functions are these revenue functions?

2. How much did Polly make from kitchen supplies in March? How much did she make from selling food products in August?

3. In which month was her revenue from kitchen supplies the greatest? The least?

4. In which month was her revenue from food products the greatest? The least?

5. Reason quantitatively. What was her total revenue from both kitchen supplies and food products in January? Explain how you arrived at your answer.

My Notes

> **MATH TIP**
>
> Some companies run their business on a fiscal year from July to June. Others, like Polly's Pasta, start the business year in January, so $t = 1$ represents January.

TECHNOLOGY TIP

You can use "Table" on your graphing calculator to quickly find the value of a function at any given x. Enter $K(t)$ as y_1 and $F(t)$ as y_2 and you can see the values for each month side by side in the table.

6. The function $S(t)$ represents Polly's total revenue from both kitchen supplies and food products. Use Polly's revenue functions to complete the table for each given value of t.

t	$K(t)$	$F(t)$	$S(t) = K(t) + F(t)$
1			
2			
3			
4			
5			

7. **Model with mathematics.** The graph below shows $K(t)$ and $F(t)$. Graph $S(t) = K(t) + F(t)$, and explain how you used the graph to find the values of $S(t)$.
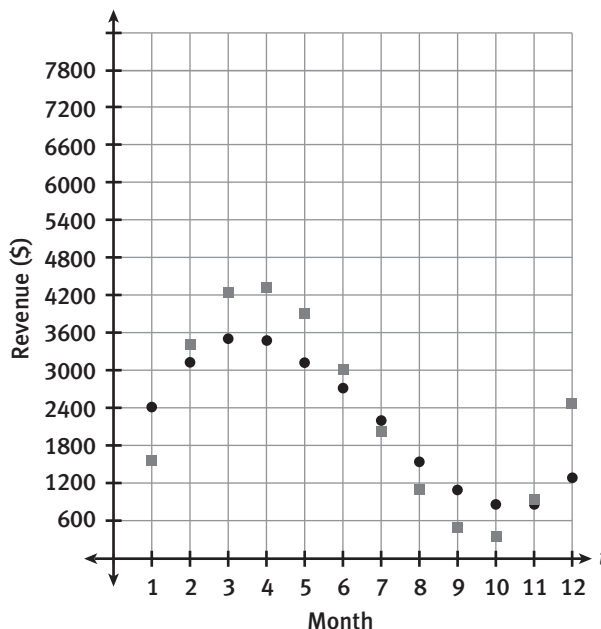
Check Your Understanding

8. Use the graph from Item 7 to approximate $S(4)$.

9. How does your answer compare to $S(4)$ from the table in Item 6?

10. **Use appropriate tools strategically.** Approximate $S(7)$ and $S(10)$ using the graph.

11. Why is the value of $S(t)$ greater than $K(t)$ and $F(t)$ for every t?

Polly's monthly operating costs are represented by the function $C(t)$, where t represents the number of the month (1–12) on the last day of the month.

$$C(t) = 5t^3 - 110t^2 + 600t + 1000$$

12. In a standard business model, profit equals total revenue minus total costs. How much profit did Patty earn in December? Explain how you found your solution.

13. Complete the table for each value of t.

t	$S(t)$	$C(t)$	$P(t) = S(t) - C(t)$
7			
8			
9			
10			
11			
12			

Check Your Understanding

14. What time frame do the values of t in the table in Item 13 represent?

15. In which month during the second half of the year did Polly's Pasta and Pizza Supply earn the least profits?

16. Reason abstractly. In any given month, would you expect the value of $P(t)$ to be greater than, less than, or equal to the value of $S(t)$ for that same month? Explain your reasoning.

17. Can you make a general statement about whether the value of $C(t)$ will be greater than, less than, or equal to the value of $P(t)$ for any given month? Use specific examples from the table in Item 13 to support your answer.

18. Reason quantitatively. Is it possible for the value of $P(t)$ to be a negative number? If so, under what circumstances?

My Notes

Most businesses study profit patterns throughout the year. This helps them make important decisions about such things as when to hire additional personnel or when to advertise more (or less).

19. Find Polly's total profit for the first quarter of the year, January–March.

20. Find Polly's total profit for the second quarter of the year, April–June.

21. Use the table in Item 13 and your answers to Items 19 and 20 to determine in which quarter Polly's Pasta and Pizza Supply earned the most profits.

LESSON 15-1 PRACTICE

22. Polly's Pasta and Pizza Supply hired a business consultant to try to reduce their operating costs. The consultant claims that if Polly implements all of his suggestions, her cost function for next year will be $C(t) = 6t^3 - 100t^2 + 400t + 900$.

  a. If the consultant is correct, how much should Polly's costs be in January of next year?

  b. How much savings is this compared to last January?

23. Use appropriate tools strategically. Use a graphing calculator to graph Polly's original and new cost functions simultaneously. Are there any months in which the consultant's plan would NOT save Polly money? If so, which months?

24. Kevin owns Kevin's Cars, and his wife Angela owns Angie's Autos. The function $K(t)$ represents the number of cars Kevin's dealership sold each month last year, and the function $A(t)$ represents the number of cars Angie's dealership sold each month. The variable t represents the number of the month (1–12) on the last day of the month.

$$K(t) = t^2 - 11t + 39$$
$$A(t) = t^2 - 7t + 28$$

  a. In January, how many cars did the two dealerships sell together?

  b. Which dealership sold more cars in June? How many more?

Learning Targets:
- Add, subtract, and multiply polynomials.
- Understand that polynomials are closed under the operations of addition, subtraction, and multiplication.

SUGGESTED LEARNING STRATEGIES: Note Taking, Marking the Text, Graphic Organizer

To add and subtract polynomials, add or subtract the coefficients of like terms.

Example A

a. Add $(3x^3 + 2x^2 - 5x + 7) + (4x^2 + 2x - 3)$.

   Step 1: Group like terms.
   $$(3x^3) + (2x^2 + 4x^2) + (-5x + 2x) + (7 - 3)$$
   Step 2: Combine like terms. $3x^3 + 6x^2 - 3x + 4$
   Solution: $3x^3 + 6x^2 - 3x + 4$

b. Subtract $(2x^3 + 8x^2 + x + 10) - (5x^2 - 4x + 6)$.

   Step 1: Distribute the negative.
   $$2x^3 + 8x^2 + x + 10 - 5x^2 + 4x - 6$$
   Step 2: Group like terms.
   $$2x^3 + (8x^2 - 5x^2) + (x + 4x) + (10 - 6)$$
   Step 3: Combine like terms. $2x^3 + 3x^2 + 5x + 4$
   Solution: $2x^3 + 3x^2 + 5x + 4$

Try These A

Find each sum or difference. Show your work.

a. $(2x^4 - 3x + 8) + (3x^3 + 5x^2 - 2x + 7)$

b. $(4x - 2x^3 + 7 - 9x^2) + (8x^2 - 6x - 7)$

c. $(3x^2 + 8x^3 - 9x) - (2x^3 + 3x - 4x^2 - 1)$

MATH TIP

Another way to group like terms is to align them vertically. For example, $(2x^2 + 6) + (4x^2 - 5x + 3)$ could be arranged like this:

$$
\begin{array}{r}
2x^2 \qquad\ +6 \\
+\ 4x^2 - 5x + 3 \\
\hline
6x^2 + x\ \ +3
\end{array}
$$

My Notes

Check Your Understanding

Find each sum or difference.

1. $(x^3 - 6x + 12) + (4x^2 + 7x - 11)$

2. $(5x^2 + 2x) - (3x^2 - 4x + 6)$

3. $(10x^3 + 2x - 5 + x^2) + (8 - 3x + x^3)$

4. What type of expression is each sum or difference above?

The *standard form of a polynomial* is $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where a is a real number and $a_n \neq 0$, with all like terms combined and written in descending order.

5. Reason abstractly and quantitatively. Use what you learned about how to add and subtract polynomials to write $S(t)$ from Item 6 and $P(t)$ from Item 12 in standard form.

MATH TIP

Multiplying polynomials looks more complicated than it is. You simply distribute each term in the first expression to each term in the second expression and then combine like terms.

6. The steps to multiply $(x + 3)(4x^2 + 6x + 7)$ are shown below. Use precise and appropriate math terminology to describe what occurs in each step.

$x(4x^2 + 6x + 7) + 3(4x^2 + 6x + 7)$	
$(4x^3 + 6x^2 + 7x) + (12x^2 + 18x + 21)$	
$4x^3 + 6x^2 + 12x^2 + 18x + 7x + 21$	
$4x^3 + 18x^2 + 25x + 21$	

My Notes

Check Your Understanding

7. Find each product. Show your work.

    a. $(x + 5)(x^2 + 4x - 5)$ **b.** $(2x^2 + 3x - 8)(2x - 3)$

    c. $(x^2 - x + 2)(x^2 + 3x - 1)$ **d.** $(x^2 - 1)(x^3 + 4x)$

8. What type of expression is each of the products in Item 7?

9. **Attend to precision.** When multiplying polynomials, how is the degree of the product related to the degrees of the factors?

LESSON 15-2 PRACTICE

For Items 10–14, perform the indicated operation. Write your answers in standard form.

10. $(x^2 + 6x - 10) - (4x^3 + 7x - 8)$

11. $(3x^2 - 2x) + (x^2 - 7x + 11)$

12. $(5x^3 + 2x - 1 + 4x^2) + (6 - 5x + x^3) - (2x^2 + 5)$

13. $(6x - 2)(x^2 + 7x - 8)$

14. $(3x^2 - 2x + 1)(x^2 + x - 4)$

15. **Critique the reasoning of others.** Marcellus made the statement that the sum of two polynomials is always a polynomial with degree equal to the highest power of x found in either of the original polynomials. He gave the following example to support his statement:

$$(3x^2 + 5x + 4) + (6x + 1) = 3x^2 + 11x + 5$$

Do you agree with Marcellus? If not, give a counterexample to support your answer.

My Notes

Learning Targets:
- Determine the quotient of two polynomials.
- Prove a polynomial identity and use it to describe numerical relationships.

> **SUGGESTED LEARNING STRATEGIES:** Note Taking, Marking the Text, Create Representations, Discussion Groups

Polynomial long division has a similar algorithm to numerical long division.

1. Use long division to find the quotient $\dfrac{592}{46}$.

Example A
Divide $x^3 - 7x^2 + 14$ by $x - 5$, using long division.

Step 1: Set up the division problem with the divisor and dividend written in descending order of degree. Include zero coefficients for any missing terms.

$$x - 5 \overline{)x^3 - 7x^2 + 0x + 14}$$

Step 2: Divide the first term of the dividend $[x^3]$ by the first term of the divisor $[x]$.

$$\begin{array}{r} x^2 \\ x-5 \overline{)x^3 - 7x^2 + 0x + 14} \end{array}$$

Step 3: Multiply the result $[x^2]$ by the divisor $[x^2(x-5) = x^3 - 5x^2]$.

$$\begin{array}{r} x^2 \\ x-5 \overline{)x^3 - 7x^2 + 0x + 14} \\ x^3 - 5x^2 \end{array}$$

Step 4: Subtract to get a new result $[-2x^2 + 0x + 14]$.

$$\begin{array}{r} x^2 \\ x-5 \overline{)x^3 - 7x^2 + 0x + 14} \\ -\left(x^3 - 5x^2\right) \\ \hline -2x^2 + 0x + 14 \end{array}$$

Step 5: Repeat the steps.

$$\begin{array}{r} x^2 - 2x - 10 \\ x-5 \overline{)x^3 - 7x^2 + 0x + 14} \\ -\left(x^3 - 5x^2\right) \\ \hline -2x^2 + 0x + 14 \\ -(-2x^2 + 10x) \\ \hline -10x + 14 \\ -(-10x + 50) \\ \hline -36 \end{array}$$

Solution: $\dfrac{x^3 - 7x^2 + 14}{x - 5} = x^3 - 2x - 10 - \dfrac{36}{x - 5}$

MATH TIP

When the division process is complete, the degree of the remainder will be less than the degree of the divisor.

Try These A

Use long division to find each quotient.

a. $(x^3 - x^2 - 6x + 18) \div (x + 3)$

b. $\dfrac{x^4 - 2x^3 - 15x^2 + 31x - 12}{x - 4}$

When a polynomial function $f(x)$ is divided by another polynomial function $d(x)$, the outcome is a new quotient function consisting of a polynomial $p(x)$ plus a remainder function $r(x)$.

$$\frac{f(x)}{d(x)} = p(x) + \frac{r(x)}{d(x)}$$

2. Follow the steps from Example A to find the quotient of $\dfrac{x^3 - x^2 + 4x + 6}{x + 2}$.

$$x + 2 \overline{)x^3 - x^2 + 4x + 6}$$

> **CONNECT** **TO** **TECHNOLOGY**
>
> You can use a CAS (computer algebra system) to perform division on more complicated quotients.

3. Find the quotient of $\dfrac{-4x^3 - 8x^2 + 32x}{x^2 + 2x - 8}$.

Check Your Understanding

Use long division to find each quotient. Show your work.

4. $(x^2 + 5x - 3) \div (x - 5)$

5. $\dfrac{4x^4 + 12x^3 + 7x^2 + x + 6}{-2x + 3}$

6. $(x^3 - 9) \div (x + 3)$

7. $\dfrac{6x^4 + 3x^3 + 13x^2 - x - 5}{3x^2 - 1}$

Synthetic division is another method of polynomial division that is useful when the divisor has the form $x - k$.

Example B

Divide $x^4 - 13x^2 + 32$ by $x - 3$ using synthetic division.

Step 1: Set up the division problem using only coefficients for the dividend and only the constant for the divisor. Include zero coefficients for any missing terms [x^3 and x].

$$3\,\rfloor\, \begin{array}{ccccc} 1 & 0 & -13 & 0 & 32 \end{array}$$

Step 2: Bring down the leading coefficient [1].

$$3\,\rfloor\, \begin{array}{ccccc} 1 & 0 & -13 & 0 & 32 \end{array}$$
$$\begin{array}{c} \downarrow \\ \hline 1 \end{array}$$

Step 3: Multiply the coefficient [1] by the divisor [3]. Write the product [$1 \cdot 3 = 3$] under the second coefficient [0] and add [$0 + 3 = 3$].

$$3\,\rfloor\, \begin{array}{ccccc} 1 & 0 & -13 & 0 & 32 \end{array}$$
$$\begin{array}{cc} & 3 \\ \hline 1 & 3 \end{array}$$

Step 4: Repeat this process until there are no more coefficients.

$$3\,\rfloor\, \begin{array}{ccccc} 1 & 0 & -13 & 0 & 32 \\ & 3 & 9 & -12 & -36 \\ \hline 1 & 3 & -4 & -12 & -4 \end{array}$$

Step 5: The numbers in the bottom row become the coefficients of the quotient. The number in the last column is the remainder. Write it over the divisor.

Solution: $x^3 + 3x^2 - 4x - 12 - \dfrac{4}{x - 3}$

My Notes

Try These B

Use long division to find each quotient.

a. $\dfrac{x^3 + 3x^2 - 10x - 24}{x + 4}$ **b.** $\dfrac{-5x^5 - 2x^4 + 32x^2 - 48x + 32}{x - 2}$

Check Your Understanding

8. Use synthetic division to divide $\dfrac{x^3 - x^2 + 4x + 6}{x + 2}$.

9. In synthetic division, how does the degree of the quotient compare to the degree of the dividend?

10. **Construct viable arguments.** Justify the following statement: The set of polynomials is closed under addition, subtraction, and multiplication, but not under division.

There are a number of polynomial identities that can be used to describe important numerical relationships in math. For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ can be used to generate a famous numerical relationship that is used in geometry.

First, let's verify the identity using what we have learned in this lesson about polynomial operations.

$$
\begin{aligned}
(x^2 + y^2)^2 &= (x^2 - y^2)^2 + (2xy)^2 \\
&= (x^2 - y^2)(x^2 - y^2) + (2xy)(2xy) \\
&= x^4 - 2x^2 y^2 + y^4 + 4x^2 y^2 \\
&= x^4 + 2x^2 y^2 + y^4 \\
&= (x^2 + y^2)^2
\end{aligned}
$$

Now that we have verified the identity, let's see how it relates to a famous numerical relationship. If we evaluate it for $x = 2$ and $y = 1$, we get:

$$(2^2 - 1^2)^2 + (2 \cdot 2 \cdot 1)^2 = (2^2 + 1^2)^2$$
$$3^2 + 4^2 = 5^2$$
$$9 + 16 = 25$$

The numbers 3, 4, and 5 are known as *Pythagorean triples* because they fit the condition $a^2 + b^2 = c^2$, which describes the lengths of the legs and hypotenuse of a right triangle.

Thus, the polynomial identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples.

Check Your Understanding

11. Use the polynomial identity above to generate a Pythagorean triple given $x = 5$ and $y = 2$.

12. Use the polynomial identity to see what happens when the values of x and y are the same. Does the identity generate a Pythagorean triple in this case? Use an example to support your answer.

13. Reason abstractly. Are there any other specific values for x and y that would not generate Pythagorean triples? If so, what value(s)?

LESSON 15-3 PRACTICE

14. Find each quotient using long division.
   a. $(6x^3 + x - 1) \div (x + 2)$

   b. $\dfrac{-2x^3 + 3x^2 - 1}{x - 1}$

15. Make sense of problems. Find each quotient using synthetic division.
   a. $(x^3 - 8) \div (x - 2)$

   b. $\dfrac{-2x^4 + 6x^3 + 3x - 1}{x - 2}$

ACTIVITY 15 PRACTICE

Write your answers on notebook paper.
Show your work.

Lesson 15-1

1. The graph below shows the number of visitors at a public library one day between the hours of 9:00 a.m. and 7:00 p.m. The round dots represent $A(t)$, the number of adult visitors, and the diamonds represent $C(t)$, the number of children and teenage visitors. Graph $V(t)$, the total number of visitors, and explain how you used the graph to find the values of $V(t)$.
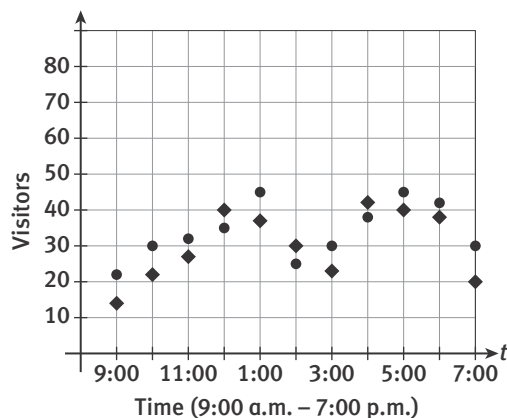
2. Examine the functions graphed in Item 1. Which of the statements is true over the given domain of the functions?
   A. $A(t) > C(t)$
   B. $C(t) > A(t)$
   C. $A(t) - C(t) > 0$
   D. $V(t) > C(t)$

3. The polynomial expressions $5x + 7$, $3x^2 + 9$, and $3x^2 - 2x$ represent the lengths of the sides of a triangle for all whole-number values of $x > 1$. Write an expression for the perimeter of the triangle.

4. In Item 3, what kind of expression is the perimeter expression?

Lesson 15-2

5. An open box will be made by cutting four squares of equal size from the corners of a 10-inch-by-12-inch rectangular piece of cardboard and then folding up the sides. The expression $V(x) = x(10 - 2x)(12 - 2x)$ can be used to represent the volume of the box. Write this expression as a polynomial in standard form.

6. Write an expression for the volume of a box that is constructed in the same way as in Item 5, but from a rectangular piece of cardboard that measures 8 inches by 14 inches. Write your expression in factored form, and then as a polynomial in standard form.

7. Write an expression to represent the combined volume of the two boxes described in Items 5 and 6.

For Items 8–13, find each sum or difference.

8. $(3x - 4) + (5x + 1)$

9. $(x^2 - 6x + 5) - (2x^2 + x + 1)$

10. $(4x^2 - 12x + 9) + (3x - 11)$

11. $(6x^2 - 13x + 4) - (8x^2 - 7x + 25)$

12. $(4x^3 + 14) + (5x^2 + x)$

13. $(2x^2 - x + 1) - (x^2 + 5x + 9)$

For Items 14–18, find each product. Write your answer as a polynomial in standard form.

14. $5x^2(4x^2 + 3x - 9)$

15. $(2x - 5)^2$

16. $(x^3 + y^3)^2$

17. $(x + 2)(3x^3 - 8x^2 + 2x - 7)$

18. $(x - 3)(2x^3 - 9x^2 + x - 6)$

Lesson 15-3

19. Which of the following quotients CANNOT be found using synthetic division?

A. $\dfrac{x^3 + 4x^2 + 5}{x^2 + 1}$

B. $\dfrac{-x^2 - x^2 + 1}{x - 1}$

C. $\dfrac{x^5 + 10}{x + 50}$

D. $\dfrac{2x^3}{x + 1}$

For Items 20–22, find each quotient using long division.

20. $\dfrac{x^2 - 6x + 4}{x + 1}$

21. $(5x^4 + 14x^3 + 9x) \div (x^2 + 3x + 1)$

22. $(2x^3 - 3x^2 + 4x - 7) \div (x - 2)$

For Items 23–25, find each quotient using synthetic division.

23. $(x^2 + 4) \div (x + 4)$

24. $\dfrac{3x^3 - 10x^2 + 12x - 22}{x - 4}$

25. $(2x^3 - 4x^2 - 15x + 4) \div (x + 3)$

MATHEMATICAL PRACTICES
Reason Abstractly and Quantitatively

26. Before answering parts a and b, review them carefully to ensure you understand all the terminology and what is being asked.
  a. When adding two polynomials, is it possible for the degree of the sum to be less than the degree of either of the polynomials being added (the addends)? If so, give an example to support your answer. If not, explain your reasoning.
  b. Is it possible for the degree of the sum to be greater than the degree of either of the addends? If so, give an example to support your answer. If not, explain your reasoning.

Binomial Theorem

Pascal's Triangle
Lesson 16-1 Introduction to Pascal's Triangle

Learning Targets:
- Find the number of combinations of an event.
- Create Pascal's triangle.

> **SUGGESTED LEARNING STRATEGIES:** Marking the Text, Vocabulary Organizer, Note Taking, Create Representations, Look for a Pattern

Many corporations, social clubs, and school classes that elect officers begin the election process by selecting a nominating committee. The responsibility of the nominating committee is to present the best-qualified nominees for the office. Mr. Darnel's class of 10 students is electing class officers. He plans to start the process by selecting a nominating committee of 4 students from the class.

Recall that in mathematics, collections of items, or in this case students, chosen without regard to order are called **combinations**. The number of combinations of n distinct things taken r at a time is denoted by $_nC_r$.

1. Use the notation above to write an expression for the number of different combinations of four-student nominating committees that Mr. Darnel could choose out of 10 students.

The formula for the number of combinations of n distinct things taken r at a time, written with **factorial** notation, is $_nC_r = \dfrac{n!}{r!(n-r)!}$.

2. Use the formula to find the number of different nominating committees that Mr. Darnel could choose.

An **alternative** notation for the number of combinations of n distinct things taken r at a time is $\begin{pmatrix} n \\ r \end{pmatrix}$.

3. Write an expression for the number of nominating committees using this notation.

My Notes

MATH TERMS

A **factorial** is the product of a natural number, n, and all natural numbers less than n, written as $n!$.

$$n! = n(n-1)(n-2) \cdot \ldots \cdot 2 \cdot 1$$

Zero factorial is defined as 1, or $0! = 1$.

ACADEMIC VOCABULARY

An **alternative** is another available possibility.

My Notes

TECHNOLOGY TIP

You can find the $_nC_r$ button with other probability functions on your graphing calculator. You can use the $_nC_r$ button to evaluate $\binom{n}{r}$ since the formulas are the same.

CONNECT TO HISTORY

Pascal's triangle is named after Blaise Pascal, a 17th-century French philosopher who made important contributions to the fields of mathematics and physics.

4. Find the values of each $_nC_r$ shown, and place them in a triangular pattern similar to the one given.

$$\binom{0}{0}$$

$$\binom{1}{0} \binom{1}{1}$$

$$\binom{2}{0} \binom{2}{1} \binom{2}{2}$$

$$\binom{3}{0} \binom{3}{1} \binom{3}{2} \binom{3}{3}$$

$$\binom{4}{0} \binom{4}{1} \binom{4}{2} \binom{4}{3} \binom{4}{4}$$

The triangular pattern that you created in Item 4 is called Pascal's triangle. Pascal's triangle has many interesting patterns.

5. What do you notice about the numbers at the end of every row in Pascal's triangle?

6. Make a conjecture about the value of $_nC_r$ when $r = 0$.

7. Make a conjecture about the value of $_nC_r$ when $r = n$.

8. **Reason quantitatively.** Starting with the second row, examine the second number in each row. Then make a conjecture about the value of $_nC_r$ when $r = 1$.

9. Write the numbers that will fill in the next row of Pascal's triangle. How did you determine what the numbers would be?

Pascal's triangle also has a number of useful applications, particularly in algebra. The best-known application is related to binomial expansion.

10. Expand each binomial using algebraic techniques.

$(a + b)^0 =$ _____

$(a + b)^1 =$ _____

$(a + b)^2 =$ _____

$(a + b)^3 =$ _____

$(a + b)^4 =$ _____

> **MATH TIP**
>
> Recall that to expand $(a + b)^2$, you must write it as a product, $(a + b)(a + b)$, and then multiply.

11. **Make use of structure.** How do the coefficients of the expanded binomials relate to the numbers in Pascal's triangle?

12. **Express regularity in repeated reasoning.** What patterns do you notice in the exponents of a and b in the expanded binomials in Item 10?

13. How does the number of terms in the expansion of $(a + b)^n$ relate to the degree n?

My Notes

Check Your Understanding

14. Explain why the order in which a teacher selects nominating committee members is *not* important.

15. In Pascal's triangle, in which row do you find the coefficients for the expansion of $(a + b)^3$? In which row do you find the coefficients for $(a + b)^4$?

16. When expanding $(a + b)^n$, which row of Pascal's triangle gives you the coefficients of the resulting polynomial?

17. Use the numbers you found in Item 9, the patterns you have observed throughout the lesson, and the conjectures you have made to expand $(a + b)^5$.

18. **Attend to precision.** What do you notice about the sum of the exponents of each term in Item 17 in comparison to the degree (n) of the binomial you expanded?

LESSON 16-1 PRACTICE

19. Evaluate $_8C_3$ and $_8C_5$.

20. Use a graphing calculator to determine how many different combinations of five-person dance committees can be selected from a class of 24 students.

21. Evaluate $_7C_2$ and $_7C_5$ without a calculator.

22. **Construct viable arguments.** Use the results of Items 19 and 21 to explain why you would expect $\binom{10}{3}$ to equal $\binom{10}{7}$. Then write a general statement of equality using n and r.

23. Write the numbers that will fill in the seventh row of Pascal's triangle.

24. Expand $(a + b)^6$.

Learning Targets:

- Know the Binomial Theorem.
- Apply the Binomial Theorem to identify the coefficients or terms of any binomial expansion.

> **SUGGESTED LEARNING STRATEGIES:** Vocabulary Organizer, Think-Pair-Share, Note Taking, Create Representations, Simplify the Problem

The **Binomial Theorem** states what we have observed about binomial expansion in the previous lesson:

> For any positive n, the binomial expansion is:
> $$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n}a^0 b^n.$$

Summation notation is a shorthand notation that can be used to represent the sum of a finite or an infinite number of terms. Here, it can be used to represent the sum of the terms in an expanded binomial.

For example, in summation notation, $(a+b)^3 = \sum_{k=0}^{3} \binom{3}{k}a^{3-k}b^k$.

MATH TERMS

Summation notation is also known as *sigma notation* because of the use of the Greek letter Σ.

1. Use the example above to write the Binomial Theorem using summation notation and $\binom{n}{k}$ to represent each coefficient.

   $(a+b)^n =$

To find the rth term of any binomial expansion $(a+b)^n$, use the expression $\binom{n}{r-1}a^{n-(r-1)}b^{r-1}$.

2. Evaluate the expression $\binom{n}{r-1}a^{n-(r-1)}b^{r-1}$ for $n=10$, $r=4$, $a=x$, and $b=3$. Simplify your answer.

My Notes

3. Find the coefficient of the sixth term in the expansion of $(x + 2)^{11}$.

4. Find the coefficient of the fourth term in the expansion of $(x - 3)^8$.

> **MATH TIP**
>
> Be careful when the binomial includes a negative number. If b is negative, for example, be sure to apply the exponent in b^{r-1} to the negative.

5. **Reason quantitatively.** Find the seventh term in the expansion of $(x + 4)^9$.

> **MATH TIP**
>
> When the binomial contains a leading coefficient other than 1, apply the exponent to the coefficient as well as the variable.

6. Find the third term in the expansion of $(2x + 3)^7$.

Check Your Understanding

7. Find the coefficient of the fourth term in the expansion of $(x + 3)^8$.

8. Find the second term in the expansion of $(x + 5)^7$.

9. Why is the coefficient of the rth term in a binomial expansion $\binom{n}{r-1}$, and not $\binom{n}{r}$?

10. **Critique the reasoning of others.** Keisha found the third term in the expansion of the binomial $(2x + 1)^4$ using the following steps: $\binom{4}{2} 2x^{4-(3-1)} 1^{3-1} = 6 \cdot 2x^2 \cdot 1^2 = 12x^2$. Do you agree or disagree with Keisha's answer? Explain your reasoning.

My Notes

11. Use the Binomial Theorem to expand each of the following binomials.
    a. $(x + 4)^7$

    b. $(x - 4)^7$

    c. $(3x + 1)^6$

Check Your Understanding

12. How many terms does the expansion of $(a + b)^{10}$ have?

13. What is the degree of $(a + b)^{10}$?

14. Use the Binomial Theorem to write the binomial expansion of:
    a. $(x + 3)^5$.
    b. $(x - 1)^8$.
    c. $(2x + 3)^4$.

15. **Construct viable arguments.** When is the Binomial Theorem useful?

My Notes

LESSON 16-2 PRACTICE

16. Write and evaluate the expression $\binom{n}{r-1}a^{n-(r-1)}b^{r-1}$ for $n = 6$, $r = 5$, $a = 2x$, and $b = 7$.

17. Find the coefficient of the third term in the expansion of $(x + 4)^5$.

18. Find the fourth term in the expansion of $(x + 3)^6$.

19. Find the second term in the expansion of $(3x - 2)^6$.

20. Use the Binomial Theorem to write the binomial expansion of $(x + 3)^4$.

21. Use the Binomial Theorem to write the binomial expansion of $(2a + 3b)^5$.

22. **Reason abstractly.** An *alternating series* is the sum of a finite or an infinite number of terms in which the signs of the terms alternate between positive and negative values. Daniel made the conjecture that when you expand $(a - b)^n$, the result will be an alternating series that always follows the pattern $+, -, +, \ldots$. Is Daniel's conjecture correct? Explain your reasoning.

ACTIVITY 16 PRACTICE

Write your answers on notebook paper.
Show your work.

Lesson 16-1

1. Which of the following would you use to find the number of different combinations of six-person nominating committees that could be chosen from a class of 25 students?

   A. $_6C_{25} = \dfrac{6!}{25!(25-6!)}$

   B. $_{25}C_6 = \dfrac{25!}{25!(25-6!)}$

   C. $_{25}C_6 = \dfrac{25!}{6!(25-6!)}$

   D. $_6C_{25} = \dfrac{6!}{6!(25-6!)}$

2. Simplify: $\dfrac{9\times8\times7\times6\times5\times4\times3\times2\times1}{(6\times5\times4\times3\times2\times1)(3\times2\times1)}$

3. Write the expression in Item 2 in $_nC_r$ notation.

4. Find the number of different combinations of four-person nominating committees that could be chosen from a class of 25 students.

5. Write the numbers that will fill in the eighth row of Pascal's triangle.

6. In which row of Pascal's triangle would you find the coefficients for the terms in the expansion of $(a+b)^{14}$?

7. Which of the following has the same value as $\begin{pmatrix} 12 \\ 7 \end{pmatrix}$?

   A. $_{12}C_7$

   B. $_{12}C_5$

   C. $\begin{pmatrix} 12 \\ 5 \end{pmatrix}$

   D. all of the above

8. Use what you have learned about the patterns in Pascal's triangle to expand $(a+b)^8$.

9. Manuela started expanding $(x+y)^9$. So far, she has written:

   $x^9 + 9x^8y + 36x^7y^2 + 84x^6y^3 + 126x^5y^4 + 126x^4y^5$

   Manuela explained to Karen that since both coefficients in the binomial are 1, the coefficients of the terms will start repeating, only backwards. Use Manuela's strategy to complete the expansion.

Lesson 16-2

10. Write $(a + b)^9$ using summation notation.

11. Write $(2x - 3)^7$ using summation notation.

12. Find the coefficient of the fourth term in the expansion of $(x + 4)^5$.

13. Which of the following is the coefficient of the third term in the expansion of $(x - 2)^7$?
   A. -84
   B. -21
   C. 21
   D. 84

14. Find the second term in the expansion of $(x + 4)^6$.

15. Find the fourth term in the expansion of $(3x - 2)^6$.

16. Use the Binomial Theorem to write the binomial expansion of $(x + 5)^4$.

17. Use the Binomial Theorem to write the binomial expansion of $(4a + b)^5$.

18. Use the Binomial Theorem to write the binomial expansion of $(x - 3)^5$.

19. Use the Binomial Theorem to write the binomial expansion of $(2x + y)^3$.

MATHEMATICAL PRACTICES
Make Sense of Problems and Persevere in Solving Them

20. Consider the statement below.

In the expansion of every binomial, the powers of x decrease by 1 from left to right when written as a polynomial in standard form.

   a. Expand the binomial $(x^2 + 1)^5$ and state whether the expansion supports or disproves the statement above and why.
   b. If the expansion disproves the statement, modify it so that it becomes a true statement.

Congruent squares of length x are cut from the corners of a 10-inch-by-15-inch piece of cardboard to create a box without a lid.

1. Write an expression in terms of x for each.
   a. the height of the box
   b. the length of the box
   c. the width of the box

2. Write a function $V(x)$ for the volume of the box in terms of x. Leave your answer in factored form.

3. Express the domain of $V(x)$ as an inequality, in interval notation, and in set notation.

4. Sketch a graph of $V(x)$ over the domain that you found in Item 3. Include the scale on each axis.

5. Use a graphing calculator to find the coordinates of the maximum point of $V(x)$ over the domain for which you graphed it. Then interpret the meaning of the maximum point.

6. Use polynomial multiplication to rewrite $V(x)$, the volume function from Item 2, as a polynomial in standard form.

7. Consider the graph of $V(x)$ over the set of real numbers. Describe the end behavior of the function using arrow notation.

8. Use long division or synthetic division to find the quotient
   $$\frac{x^5 + 3x^3 - 4x^2 + 2x + 6}{x - 1}.$$

9. Draw the first six rows of Pascal's triangle. Then use the triangle to expand $(a + b)^5$.

10. Use the Binomial Theorem to expand $(2x + 1)^6$.

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
	The solution demonstrates these characteristics:			
Mathematics Knowledge and Thinking (Items 5-10)	• Effective understanding and identification of key features of polynomial functions including extreme values and end behavior • Clear and accurate understanding of operations with polynomials (multiplication, division, binomial expansion)	• A functional understanding and accurate identification of key features of polynomial functions including extreme values and end behavior • Largely correct understanding of operations with polynomials (multiplication, division, binomial expansion)	• Partial understanding and partially accurate identification of key features of polynomial functions including extreme values and end behavior • Partially correct operations with polynomials (multiplication, division, binomial expansion)	• Little or no understanding and inaccurate identification of key features of polynomial functions including extreme values and end behavior • Incomplete or mostly inaccurate operations with polynomials (multiplication, division, binomial expansion)
Problem Solving (Item 5)	• An appropriate and efficient strategy that results in a correct answer	• A strategy that may include unnecessary steps but results in a correct answer	• A strategy that results in some incorrect answers	• No clear strategy when solving problems
Mathematical Modeling / Representations (Items 1-5)	• Fluency in creating polynomial expressions and functions to model real-world scenarios, including reasonable domain • Clear and accurate understanding of how to graph and identify features of polynomial functions by hand and using technology and represent intervals using inequalities, interval notation, and set notation	• Little difficulty in creating polynomial expressions and functions to model real-world scenarios, including reasonable domain • Mostly accurate understanding of how to graph and identify features of polynomial functions by hand and using technology and represent intervals using inequalities, interval notation, and set notation	• Partial understanding of creating polynomial expressions and functions to model real-world scenarios, including reasonable domain • Partial understanding of how to graph and identify features of polynomial functions by hand and using technology and represent intervals using inequalities, interval notation, and set notation	• Little or no understanding of creating polynomial expressions and functions to model real-world scenarios, including reasonable domain • Inaccurate or incomplete understanding of how to graph and identify features of polynomial functions by hand and using technology and represent intervals using inequalities, interval notation, and set notation
Reasoning and Communication (Items 5, 7)	• Precise use of appropriate math terms and language to explain the maximum point in terms of a real-world scenario • Clear and accurate explanation of the end behavior of a polynomial function	• Adequate explanation of the maximum point in terms of a real-world scenario • Adequate explanation of the end behavior of a polynomial function	• Misleading or confusing explanation of the maximum point in terms of a real-world scenario • Misleading or confusing explanation of the end behavior of a polynomial function	• Incomplete or inadequate explanation of the maximum point in terms of a real-world scenario • Incomplete or inaccurate explanation of the end behavior of a polynomial function

Factors of Polynomials

How Many Roots?
Lesson 17-1 Algebraic Methods

Learning Targets:

- Determine the linear factors of polynomial functions using algebraic methods.
- Determine the linear or quadratic factors of polynomials by factoring the sum or difference of two cubes and factoring by grouping.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Note Taking, Look for a Pattern, Simplify a Problem, Identify a Subtask

When you factor a polynomial, you rewrite the original polynomial as a product of two or more polynomial factors.

1. State the common factor of the terms in the polynomial $4x^3 + 2x^2 - 6x$. Then factor the polynomial.

2. **Make use of structure.** Consider the expression $x^2(x - 3) + 2x(x - 3) + 3(x - 3)$.

   a. How many terms does it have?

   b. What factor do all the terms have in common?

3. Factor $x^2(x - 3) + 2x(x - 3) - 3(x - 3)$.

Some quadratic trinomials, $ax^2 + bx + c$, can be factored into two binomial factors.

Example A

Factor $2x^2 + 7x - 4$.

Step 1:	Find the product of a and c.	$2(-4) = -8$
Step 2:	Find the factors of ac that have a sum of b, 7.	$8 + (-1) = 7$
Step 3:	Rewrite the polynomial, separating the linear term.	$2x^2 + 8x - 1x - 4$
Step 4:	Group the first two terms and the last two terms.	$(2x^2 + 8x) + (-x - 4)$
Step 5:	Factor each group separately.	$2x(x + 4) - 1(x + 4)$
Step 6:	Factor out the binomial.	$(x + 4)(2x - 1)$

Solution: $2x^2 + 7x - 4 = (x + 4)(2x - 1)$

MATH TIP

Check your answer to a factoring problem by multiplying the factors together to get the original polynomial.

My Notes

My Notes

Try These A
a. Use Example A as a guide to factor $6x^2 + 19x + 10$. Show your work.

Factor each trinomial. Show your work.

b. $3x^2 - 8x - 3$
c. $2x^2 + 7x + 6$

Some higher-degree polynomials can also be *factored by grouping*.

Example B
a. Factor $3x^3 + 9x^2 + 4x + 12$ by grouping.

Step 1:	Group the terms.	$(3x^3 + 9x^2) + (4x + 12)$
Step 2:	Factor each group separately.	$3x^2(x + 3) + 4(x + 3)$
Step 3:	Factor out the binomial.	$(x + 3)(3x^2 + 4)$

Solution: $3x^3 + 9x^2 + 4x + 12 = (x + 3)(3x^2 + 4)$

b. Factor $3x^4 + 9x^3 + 4x + 12$ by grouping.

Step 1:	Group the terms.	$(3x^4 + 9x^3) + (4x + 12)$
Step 2:	Factor each group separately.	$3x^3(x + 3) + 4(x + 3)$
Step 3:	Factor out the binomial.	$(x + 3)(3x^3 + 4)$

Solution: $3x^4 + 9x^3 + 4x + 12 = (x + 3)(3x^3 + 4)$

Try These B
Factor by grouping. Show your work.

a. $2x^3 + 10x^2 - 3x - 15$
b. $4x^4 + 7x^3 + 4x + 7$

My Notes

Check Your Understanding

4. Factor $7x^4 + 21x^3 - 14x^2$.

5. Factor $6x^2 + 11x + 4$.

6. Factor by grouping.
   a. $8x^3 - 64x^2 + x - 8$
   b. $12x^4 + 2x^3 - 30x - 5$

7. **Reason abstractly.** What is the purpose of separating the linear term in a quadratic trinomial when factoring?

A difference of two squares can be factored by using a specific pattern, $a^2 - b^2 = (a + b)(a - b)$. A *difference of two cubes* and a *sum of two cubes* also have a factoring pattern.

Difference of Cubes	**Sum of Cubes**
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

8. What patterns do you notice in the formulas that appear above?

9. **Express regularity in repeated reasoning.** Factor each difference or sum of cubes.
   a. $x^3 - 8$ **b.** $x^3 + 27$

   c. $8x^3 - 64$ **d.** $27 + 125x^3$

MATH TIP

It is a good strategy to first identify and label a and b. This makes it easier to substitute into the formula.

Some higher-degree polynomials can be factored by using the same patterns or formulas that you used when factoring quadratic binomials or trinomials.

10. Use the difference of squares formula $a^2 - b^2 = (a + b)(a - b)$ to factor $16x^4 - 25$. (It may help to write each term as a square.)

11. Reason quantitatively. Explain the steps used to factor $2x^5 + 6x^3 - 8x$.

$2x^5 + 6x^3 - 8x$	Original expression
$= 2x(x^4 + 3x^2 - 4)$	
$= 2x(x^2 + 4)(x^2 - 1)$	
$= 2x(x^2 + 4)(x + 1)(x - 1)$	

12. Use the formulas for quadratic binomials and trinomials to factor each expression.

a. $x^4 + x^2 - 20$

b. $16x^4 - 81$

c. $(x - 2)^4 + 10(x - 2)^2 + 9$

Check Your Understanding

13. Factor each difference or sum of cubes.
   a. $125x^3 + 216$
   b. $x^6 - 27$

14. Use the formulas for factoring quadratic binomials and trinomials to factor each expression.
   a. $x^4 - 14x^2 + 33$
   b. $81x^4 - 625$

15. Attend to precision. A *linear factor* is a factor that has degree 1. A *quadratic factor* has degree 2. The factored expression in Item 11 has 3 linear factors and 1 quadratic factor. What is true about the degree of the factors in relation to the degree of the original expression?

LESSON 17-1 PRACTICE

Factor each expression.

16. $3x^2 - 14x - 5$

17. $8x^3 + 27$

18. $x^4 - 5x^2 - 36$

19. $2x^4 - x^3 - 18x^2 + 9x$

20. Model with mathematics. The trinomial $4x^2 + 12x + 9$ represents the area of a square. Write an expression that represents the length of one side of the square. Explain your answer.

Learning Targets:
- Know and apply the Fundamental Theorem of Algebra.
- Write polynomial functions, given their degree and roots.

SUGGESTED LEARNING STRATEGIES: Vocabulary Organizer, Note Taking, Graphic Organizer, Work Backward

As a consequence of the ***Fundamental Theorem of Algebra***, a polynomial $p(x)$ of degree $n \geq 0$ has exactly n linear factors, counting factors used more than once.

MATH TERMS

Let $p(x)$ be a polynomial function of degree n, where $n > 0$. The **Fundamental Theorem of Algebra** states that $p(x) = 0$ has at least one zero in the complex number system.

Example A

Find the zeros of $f(x) = 3x^3 + 2x^2 + 6x + 4$. Show that the Fundamental Theorem of Algebra is true for this function by counting the number of zeros.

Step 1: Set the function equal to 0. $\qquad 3x^3 + 2x^2 + 6x + 4 = 0$

Step 2: Look for a factor common to all terms, use the quadratic trinomial formulas, or factor by grouping, as was done here. $\qquad (3x^3 + 6x) + (2x^2 + 4) = 0$

Step 3: Factor each group separately. $\qquad 3x(x^2 + 2) + 2(x^2 + 2) = 0$

Step 4: Factor out the binomial to write the factors. $\qquad (x^2 + 2)(3x + 2) = 0$

Step 5: Use the Zero Product Property to solve for x.

$$x^2 + 2 = 0 \qquad 3x + 2 = 0$$
$$x = \pm i\sqrt{2} \qquad x = -\frac{2}{3}$$

Solution: $x = \pm i\sqrt{2}; \; x = -\frac{2}{3}$.

All three zeros are in the complex number system.

MATH TIP

When counting the number of zeros, remember that when solutions have the $\pm$ symbol, such as $\pm a$, this represents two different zeros, a and $-a$.

Try These A

Find the zeros of the functions by factoring and using the Zero Product Property. Show that the Fundamental Theorem of Algebra is true for each function by counting the number of complex zeros.

a. $f(x) = x^3 + 9x$

b. $g(x) = x^4 - 16$

MATH TIP

All real numbers are complex numbers with an imaginary part of zero.

c. $h(x) = (x - 2)^2 + 4(x - 2) + 4$

d. $p(x) = x^3 - 64$

e. $k(x) = x^3 + 5x^2 + 9x + 45$

f. $w(x) = x^3 + 216$

MATH TIP

When you factor the sum or difference of cubes, the result is a linear factor and a quadratic factor. To find the zeros of the quadratic factor, use the quadratic formula.

My Notes

Check Your Understanding

For Items 1–4, find the zeros of the functions. Show that the Fundamental Theorem of Algebra is true for each function by counting the number of complex zeros.

1. $g(x) = x^4 - 81$

2. $h(x) = x^3 + 8$

3. $f(x) = x^4 + 25x^2$

4. $k(x) = x^3 - 7x^2 + 4x - 28$

5. **Make use of structure.** As a consequence of the Fundamental Theorem of Algebra, how many linear factors, including multiple factors, does the function
$f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ have?

6. What is the minimum number of real zeros for the function in Item 4? Explain your reasoning.

7. Create a flowchart, other organizational scheme, or set of directions for finding the zeros of polynomials.

It is possible to find a polynomial function, given its zeros.

Example B
Find a polynomial function of 3rd degree that has zeros 0, 2, and -3.

Step 1: Write the factors. $\qquad\qquad f(x) = (x)(x - 2)(x + 3)$

Step 2: Multiply the binomials. $\qquad\quad f(x) = (x)(x^2 + x - 6)$

Step 3: Distribute the x. $\qquad\qquad\quad f(x) = x^3 + x^2 - 6x$

Solution: $f(x) = x^3 + x^2 - 6x$

Try These B
Find a polynomial function with the indicated degree and zeros.

a. $n = 3$; zeros 0, 5, -7

b. $n = 4$; zeros $\pm 1, \pm 5$

MATH TIP

If a is a zero of a polynomial function, then $(x - a)$ is a factor of the polynomial.

My Notes

The *Complex Conjugate Root Theorem* states that if $a + bi$, $b \neq 0$, is a zero of a polynomial function with real coefficients, the conjugate $a - bi$ is also a zero of the function.

Example C

a. Find a polynomial function of 3rd degree that has zeros 3 and $4i$.

Step 1: Use the Complex Conjugate Root Theorem to find all zeros. $\qquad x = 3, x = 4i, x = -4i$

Step 2: Write the factors. $\qquad f(x) = (x - 3)(x - 4i)(x + 4i)$

Step 3: Multiply the factors that contain i. $\qquad f(x) = (x - 3)(x^2 + 16)$

Step 4: Multiply out the factors to get the polynomial function. $\qquad f(x) = x^3 - 3x^2 + 16x - 48$

Solution: $f(x) = x^3 - 3x^2 + 16x - 48$

b. Find a polynomial function of 4th degree that has zeros 1, -1, and $1 + 2i$.

Step 1: Use the Complex Conjugate Root Theorem to find all zeros. $\quad x = 1, x = -1, x = 1 + 2i, x = 1 - 2i$

Step 2: Write the factors. $\qquad f(x) = (x - 1)(x + 1)$
$\qquad\qquad\qquad\qquad\qquad (x - (1 + 2i))(x - (1 - 2i))$

Step 3: Multiply using the fact that $(a - b)(a + b) = a^2 - b^2$. $\qquad f(x) = (x^2 - 1)(x^2 - 2x + 5)$

Step 4: Multiply out the factors to get the polynomial function. $\qquad f(x) = x^4 - 2x^3 + 4x^2 + 2x - 5$

Solution: $f(x) = x^4 - 2x^3 + 4x^2 + 2x - 5$

Try These C

Reason quantitatively. Write a polynomial function of nth degree that has the given real or complex roots.

a. $n = 3$; $x = -2$, $x = 3i$

b. $n = 4$; $x = 3$, $x = -3$, $x = 1 + 2i$

c. $n = 4$; $x = 2$, $x = -5$, and $x = -4$ is a double root

My Notes

Check Your Understanding

8. **Reason abstractly.** If $3 + 2i$ is a zero of $p(x)$, what is another zero of $p(x)$?

For Items 9–12, write a polynomial function of nth degree that has the given real or complex roots.

9. $n = 3; x = -2, x = 0, x = 5$

10. $n = 3; x = 3, x = 2i$

11. $n = 4; x = 5, x = -5, x = i$

12. $n = 4; x = 3, x = -4$, and $x = 2$ is a double root

LESSON 17-2 PRACTICE

For Items 13–15, find the zeros of the functions. Show that the Fundamental Theorem of Algebra is true for each function by counting the number of complex zeros.

13. $f(x) = x^3 + 1000$

14. $f(x) = x^3 - 4x^2 + 25x - 100$

15. $f(x) = x^4 - 3x^3 + x^2 - 3x$

For Items 16–19, write a polynomial function of nth degree that has the given real or complex zeros.

16. $n = 3; x = 9, x = 2i$

17. $n = 3; x = -1, x = 4 + i$

18. $n = 4; x = -6$ is a double zero and $x = 2$ is a double zero

19. **Construct viable arguments.** Use the theorems you have learned in this lesson to determine the degree of a polynomial that has zeros $x = 3, x = 2i$, and $x = 4 + i$. Justify your answer.

ACTIVITY 17 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 17-1

1. State the common factor of the terms in the polynomial $5x^3 + 30x^2 - 10x$. Then factor the polynomial.

2. Which of the following is one of the factors of the polynomial $15x^2 - x - 2$?

   A. $x - 2$
   B. $5x - 2$
   C. $5x + 1$
   D. $3x - 1$

3. Factor each polynomial.
   a. $6x^2 + 7x - 5$
   b. $14x^2 + 25x + 6$

4. Factor by grouping.
   a. $8x^3 - 64x^2 + x - 8$
   b. $12x^4 + 2x^3 - 30x - 5$

5. Factor each difference or sum of cubes.
   a. $125x^3 + 216$
   b. $x^6 - 27$

6. Use the formulas for factoring quadratic binomials and trinomials to factor each expression.
   a. $x^4 - 14x^2 + 33$
   b. $81x^4 - 625$
   c. $x^4 + 17x^2 + 60$
   d. $x^6 - 100$

Lesson 17-2

7. Which theorem states that a polynomial of degree n has exactly n linear factors, counting multiple factors?
   A. Binomial Theorem
   B. Quadratic Formula
   C. Fundamental Theorem of Algebra
   D. Complex Conjugate Root Theorem

8. Find the zeros of the functions by factoring and using the Zero Product Property. Identify any multiple zeros.
   a. $f(x) = 2x^4 + 18x^2$
   b. $g(x) = 3x^3 - 3$
   c. $h(x) = 5x^3 - 6x^2 - 45x + 54$
   d. $h(x) = 3x^4 - 36x^3 + 108x^2$

9. The table of values shows coordinate pairs on the graph of $f(x)$. Which of the following could be $f(x)$?
   A. $x(x + 1)(x - 1)$
   B. $(x - 1)(x + 1)(x - 3)$
   C. $(x + 1)^2(x + 3)$
   D. $(x + 1)(x - 2)^2$

x	$f(x)$
-1	0
0	3
1	0
2	-3

10. Write a polynomial function of nth degree that has the given zeros.
    a. $n = 3$; $x = 1$, $x = 6$, $x = -6$
    b. $n = 4$; $x = -3$, $x = 3$, $x = 0$, $x = 4$

11. Which of the following polynomial functions has multiple roots at $x = 0$?
    A. $f(x) = x^2 - x$
    B. $f(x) = x^3 - x^2$
    C. $f(x) = x^3 - x$
    D. all of the above

12. Write a polynomial function of nth degree that has the given real or complex roots.
    a. $n = 3$; $x = -2$, $x = 5$, $x = -5$
    b. $n = 4$; $x = -3$, $x = 3$, $x = 5i$
    c. $n = 3$; $x = -2$, $x = 1 + 2i$

13. Give the degree of the polynomial function with the given real or complex roots.
    a. $x = -7$, $x = 1$, $x = 4i$
    b. $x = -2$, $x = 2$, $x = 0$, $x = 4 + i$
    c. $x = 2i$, $x = 1 - 3i$

14. Which of the following could be the factored form of the polynomial function
    $f(x) = x^4 + \ldots + 48$?
    I. $f(x) = (x + 1)(x + 3)(x + 4i)(x - 4i)$
    II. $f(x) = (x + 2)^2(x - 1)(x + 4)(x - 6)$
    III. $f(x) = (x + 3)(x - 8)(x + 2i)(x - 2i)$
    A. I only
    B. I and II only
    C. II only
    D. I, II, and III

15. Explain your reason(s) for eliminating each of the polynomials you did not choose in Item 14.

MATHEMATICAL PRACTICES
Use Appropriate Tools Strategically

16. Use the information below to write a polynomial function, first in factored form and then in standard form.

    Fact: The graph only touches the x-axis at a double zero; it does not cross through the axis.

    Clue: One of the factors of the polynomial is $(x + i)$.

Graphs of Polynomials

Getting to the End Behavior
Lesson 18-1 Graphing Polynomial Functions

Learning Targets:

- Graph polynomial functions by hand or using technology, identifying zeros when suitable factorizations are available, and showing end behavior.
- Recognize even and odd functions from their algebraic expressions.

SUGGESTED LEARNING STRATEGIES: Look for a Pattern, Create Representations, Think-Pair-Share, Vocabulary Organizer, Marking the Text

1. **Make sense of problems.** Each graph below shows a polynomial of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where $a_n \neq 0$. Apply what you know about graphs of polynomials to match each graph to one of the equations below. Write the equation under the graph. Justify your answers.

$$y = -2x^3 - 4x^2 + 1$$
$$y = 2x^3 - 4x^2 + 1$$
$$y = -3x^4 + 8x^2 + 1$$
$$y = 3x^4 - 8x^2 + 1$$
$$y = -2x^5 - 4x^4 + 5x^3 + 8x^2 - 5x$$
$$y = 2x^5 + 4x^4 - 5x^3 - 8x^2 + 5x$$

a.   b.   c.

d.   e.   f.

My Notes

Polynomials can be written in factored form or in standard form. Each form provides useful clues about how the graph will behave. Work on Item 2 with your group. As needed, refer to the Glossary to review translations of key terms. Incorporate your understanding into group discussions to confirm your knowledge and use of key mathematical language.

2. **Model with mathematics.** Sketch a graph of each function. For graphs **b** through **e**, identify the information revealed by the unfactored polynomial compared to the factored polynomial.
   a. $f(x) = x + 3$
   b. $g(x) = x^2 - 9 = (x + 3)(x - 3)$

   c. $h(x) = x^3 + x^2 - 9x - 9 = (x + 3)(x - 3)(x + 1)$

   d. $k(x) = x^4 - 10x^2 + 9 = (x + 3)(x - 3)(x + 1)(x - 1)$

   e. $p(x) = x^5 + 10x^4 + 37x^3 + 60x^2 + 36x = x(x + 2)^2(x + 3)^2$

Check Your Understanding

3. Sketch a graph of the cubic function:
   $p(x) = x^3 - 2x^2 - 19x + 20 = (x + 4)(x - 1)(x - 5)$

4. Identify the information revealed by the unfactored polynomial in Item 3 compared to the factored polynomial.

5. Use your calculator to graph the function in Item 3.

6. Compare the calculator image with your sketch. What information is not revealed by either the standard form or factored form of a polynomial?
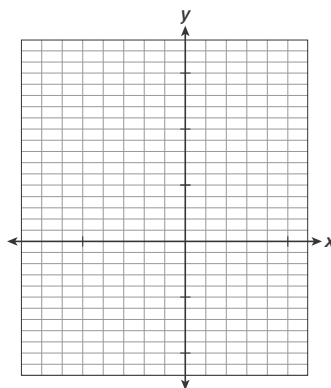
My Notes

Polynomial functions are *continuous functions*, meaning that their graphs have no gaps or breaks. Their graphs are smooth, unbroken curves with no sharp turns. Graphs of polynomial functions with degree n have n zeros (real number zeros are x-intercepts), as you saw in the Fundamental Theorem of Algebra. They also have at most $n - 1$ *relative extrema*.

7. Find the x-intercepts of $f(x) = x^4 + 3x^3 - x^2 - 3x$.

8. Find the y-intercept of $f(x)$.

9. Reason quantitatively. How can the zeros of a polynomial function help you identify where the relative extrema will occur?

10. The relative extrema of the function $f(x) = x^4 + 3x^3 - x^2 - 3x$ occur at approximately $x = 0.6$, $x = -0.5$, and $x = -2.3$. Use these x-values to find the approximate values of the extrema and graph the function.

11. Sketch a graph of $f(x) = -x^3 - x^2 - 6x$.

12. Sketch a graph of $f(x) = x^4 - 10x^2 + 9$.

MATH TERMS

Maxima and *minima* are known as **extrema**. They are the greatest value (the maximum) or the least value (the minimum) of a function over an interval or the entire domain.

When referring to extrema that occur within a specific interval of the domain, they are called **relative extrema**.

When referring to values that are extrema for the entire domain of the function, they are called **global extrema**.

CONNECT TO AP

In calculus, you will use the first derivative of a polynomial function to algebraically determine the coordinates of the extrema.

My Notes

Check Your Understanding

13. **Use appropriate tools strategically.** Use a graphing calculator to graph the polynomial functions. Verify that their x- and y-intercepts are correct, and determine the coordinates of the relative extrema.
    a. $f(x) = x^3 + 7x^2 - x - 7$
    b. $h(x) = x^4 - 13x^2 + 36$

14. What is the maximum number of relative extrema a fifth-degree polynomial function can have?

15. **Construct viable arguments.** Explain why relative extrema occur between the zeros of a polynomial function.

LESSON 18-1 PRACTICE

16. Sketch the graph of a polynomial function that decreases as $x \to \pm\infty$ and has zeros at $x = -10, -3, 1,$ and 4.

17. Sketch a graph of $f(x)$ given below. Identify the information revealed by the unfactored polynomial compared to the factored polynomial.
    $f(x) = x^5 - 2x^4 - 25x^3 + 26x^2 + 120x = x(x-5)(x-3)(x+2)(x+4)$

18. Use a graphing calculator to graph $f(x) = x^3 - x^2 - 49x + 49$.

19. Find all intercepts of the function in Item 18.

20. Find the relative maximum and minimum values of the function in Item 18.

21. **Make sense of problems.** A fourth-degree even polynomial function has a relative maximum at $(0, 5)$ and relative minimums at $(-4, 1)$ and $(4, 1)$. How many real zeros does this function have? Explain your reasoning.

Learning Targets:

- Know and apply the Rational Root Theorem and Descartes' Rule of Signs.
- Know and apply the Remainder Theorem and the Factor Theorem.

> **SUGGESTED LEARNING STRATEGIES:** Shared Reading, Vocabulary Organizer, Marking the Text, Note Taking

Some polynomial functions, such as $f(x) = x^3 - 2x^2 - 5x + 6$, are not factorable using the tools that you have. However, it is still possible to graph these functions without a calculator. The following tools will be helpful.

Rational Root Theorem	Finds possible rational roots
Descartes' Rule of Signs	Finds the possible number of real roots
Remainder Theorem	Determines if a value is a zero
Factor Theorem	Another way to determine if a value is a zero

> **Rational Root Theorem**
>
> If a polynomial function $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$,
> $a_n \neq 0$, has integer coefficients, then every rational root of $f(x) = 0$ has
> the form $\dfrac{p}{q}$, where p is a factor of a_0, and q is a factor of a_n.

The Rational Root Theorem determines the possible rational roots of a polynomial.

1. Consider the quadratic equation $2x^2 + 9x - 3 = 0$.
   a. Make a list of the only possible rational roots to this equation.

   b. **Reason abstractly.** Explain why you think these are the only possible rational roots.

   c. Does your list of rational roots satisfy the equation?

My Notes

2. What can you conclude from Item 1 part c?

3. Reason quantitatively. Verify your conclusion in Item 1 part c by finding the roots of the equation in Item 1 using the Quadratic Formula. Show your work.

Example A

Find all the possible rational roots of $f(x) = x^3 - 2x^2 - 5x + 6$.

Step 1: Find the factors q of the leading coefficient 1 and the factors p of the constant term 6.

q could equal ± 1
p could equal $\pm 1, \pm 2, \pm 3, \pm 6$

Step 2: Write all combinations of $\dfrac{p}{q}$. Then simplify.

$\dfrac{\pm 1, \pm 2, \pm 3, \pm 6}{\pm 1}$

Solution: $\pm 1, \pm 2, \pm 3, \pm 6$

Try These A

Find all the possible rational roots of $f(x) = 2x^3 + 7x^2 + 2x - 3$.

The Rational Root Theorem can yield a large number of possible roots. To help eliminate some possibilities, you can use Descartes' Rule of Signs. While Descartes' rule does not tell you the value of the roots, it does tell you the maximum number of positive and negative real roots.

Descartes' Rule of Signs

If $f(x)$ is a polynomial function with real coefficients and a nonzero constant term arranged in descending powers of the variable, then

- the number of positive real roots of $f(x) = 0$ equals the number of variations in sign of the terms of $f(x)$, or is less than this number by an even integer.
- the number of negative real roots of $f(x) = 0$ equals the number of variations in sign of the terms of $f(-x)$, or is less than this number by an even integer.

Example B

Find the number of positive and negative roots of $f(x) = x^3 - 2x^2 - 5x + 6$.

Step 1: Determine the sign changes in $f(x)$: $f(x) = x^3 - 2x^2 - 5x + 6$
There are two sign changes:
- one between the first and second terms when the sign goes from positive to negative
- one between the third and fourth terms when the sign goes from negative to positive

So, there are either two or zero positive real roots.

Step 2: Determine the sign changes in $f(-x)$: $f(-x) = -x^3 - 2x^2 + 5x + 6$
There is one sign change:
- between the second and third terms when the sign goes from negative to positive

So, there is one negative real root.

Solution: There are either two or zero positive real roots and one negative real root.

Try These B

Find the number of positive and negative roots of $f(x) = 2x^3 + 7x^2 + 2x - 3$.

Check Your Understanding

4. Determine all the possible rational roots of $f(x) = 2x^3 - 2x^2 - 4x + 5$.

5. Determine the possible number of positive and negative real zeros for $h(x) = x^3 - 4x^2 + x + 5$.

6. The function $f(x) = x^3 + x^2 + x + 1$ has only one possible rational root. What is it? Explain your reasoning.

7. **Construct viable arguments.** Explain the circumstances under which the only possible rational roots of a polynomial are integers.

You have found all the possible rational roots and the number of positive and negative real roots for the function $f(x) = x^3 - 2x^2 - 5x + 6$. The next two theorems will help you find the zeros of the function. The Remainder Theorem tells if a possible root is actually a zero or just another point on the graph of the polynomial. The Factor Theorem gives another way to test if a possible root is a zero.

My Notes

Remainder Theorem

If a polynomial $P(x)$ is divided by $(x - k)$, where k is a constant, then the remainder r is $P(k)$.

Factor Theorem

A polynomial $P(x)$ has a factor $(x - k)$ if and only if $P(k) = 0$.

Example C

Use synthetic division to find the zeros and factor $f(x) = x^3 - 2x^2 - 5x + 6$.

From Examples A and B, you know the possible rational zeros are $\pm 1, \pm 2, \pm 3, \pm 6$. You also know that the polynomial has either two or zero positive real roots and one negative real root. Now it is time to check each of the possible rational roots to determine if they are zeros of the function.

Step 1: Divide $(x^3 - 2x^2 - 5x + 6)$ by $(x + 1)$.

$$
\begin{array}{r|rrrr}
-1 & 1 & -2 & -5 & 6 \\
   &   & -1 & 3 & 2 \\
\hline
   & 1 & -3 & -2 & 8
\end{array}
$$
So, you have found a point, $(-1, 8)$.

Step 2: Continue this process, finding either points on the polynomial and/or zeros for each of the possible roots.

Divide $(x^3 - 2x^2 - 5x + 6)$ by $(x - 1)$.

$$
\begin{array}{r|rrrr}
1 & 1 & -2 & -5 & 6 \\
  &   & 1 & -1 & -6 \\
\hline
  & 1 & -1 & -6 & 0
\end{array}
$$
So, you have found a zero, $(1, 0)$, and a factor, $f(x) = (x - 1)(x^2 - x - 6)$.

Step 3: As soon as you have a quadratic factor remaining after the division process, you can factor the quadratic factor by inspection, if possible, or use the Quadratic Formula.

Solution: $f(x) = (x - 1)(x + 2)(x - 3)$; The real zeros are $1, -2$, and 3.

Try These C

Use synthetic division and what you know from Try These A and B to find the zeros and factor $f(x) = 2x^3 + 7x^2 + 2x - 3$.
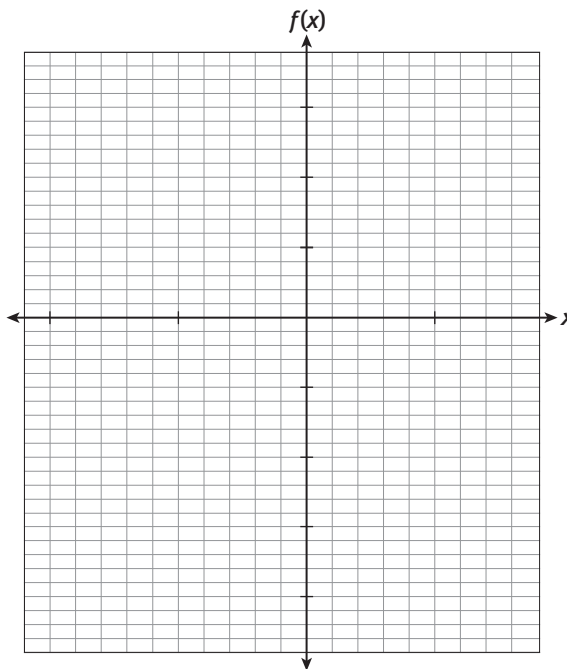
Check Your Understanding

8. One of the possible rational roots of $f(x) = x^3 - 2x^2 - 4x + 5$ is 5. If you divide $x^3 - 2x^2 - 4x + 5$ by $x - 5$, the remainder is 60. What information does this give you about the graph of $f(x)$?

9. If a polynomial $P(x)$ is divided by $(x - k)$ and the remainder is 0, what does this tell you about the value k?

Using the Factor Theorem, follow a similar process to find the real zeros.

Example D

Use the Factor Theorem to find the real zeros of $f(x) = x^3 - 2x^2 - 5x + 6$.
Again, you know the possible rational roots are $\pm 1, \pm 2, \pm 3, \pm 6$.

Step 1: Test $(x + 1)$: $f(-1) = (-1)^3 - 2(-1)^2 - 5(-1) + 6 = 8$
So, you have a point, $(-1, 8)$.

Step 2: Test $(x - 1)$: $f(1) = (1)^3 - 2(1)^2 - 5(1) + 6 = 0$
So, you have a zero at $x = 1$.

Step 3: Test $(x - 2)$: $f(2) = (2)^3 - 2(2)^2 - 5(2) + 6 = -4$
So, you have a point, $(2, -4)$.

Step 4: Continue to test rational zeros or use division to simplify the polynomial and factor or use the Quadratic Formula to find the real zeros.

Solution: The real zeros are $1, -2,$ and 3.

Try These D

Use the Factor Theorem and what you know from Try These A and B to find the real zeros of $f(x) = 2x^3 + 7x^2 + 2x - 3$.

Example E

Graph $f(x) = x^3 - 2x^2 - 5x + 6$ using the information you found in Examples A–D. Also include the y-intercept and what you know about the end behavior of the function.

x	f(x)
−1	8
1	0
−2	0
0	6
3	0
2	−4

My Notes

Try These E
Graph $f(x) = 2x^3 + 7x^2 + 2x - 3$ using the information from Try These A–D. Include a scale on both axes.

Check Your Understanding

10. The possible rational roots of $g(x) = 2x^4 + 5x^3 - x^2 + 5x - 1$ are $\pm\frac{1}{2}$ and ± 1. List the possible factors of $g(x)$.

11. For the function $f(x) = x^3 - 2x^2 - 4x + 5$, $f(-1) = 6$. Is $(x + 1)$ a factor of $f(x)$? Explain your reasoning.

12. For the function $p(x) = x^3 - 2x^2 - 4x + 8$, $p(2) = 0$. Name one factor of $f(x)$.

LESSON 18-2 PRACTICE

13. Determine all the possible rational roots of $f(x) = x^3 - 5x^2 - 17x + 21$.

14. Use the Remainder Theorem to determine which of the possible rational roots for the function in Item 13 are zeros of the function.

15. Use the information from Item 14 to graph the function in Item 13.

16. Determine the possible number of positive and negative real roots for $h(x) = 2x^3 + x^2 - 5x + 2$.

17. **Model with mathematics.** Graph $h(x) = 2x^3 + x^2 - 5x + 2$.

18. **Reason quantitatively.** Use the Rational Root Theorem to write a fourth-degree polynomial function that has possible rational roots of $\pm\frac{1}{4}, \pm\frac{7}{4}, \pm 1, \pm 7$. Then use Descartes' Rule of Signs to modify your answer to ensure that none of the actual zeros are positive rational numbers.

Learning Targets:
- Compare properties of two functions each represented in a different way.
- Solve polynomial inequalities by graphing.

> **SUGGESTED LEARNING STRATEGIES:** Create Representations, Note Taking, Marking the Text, Identify a Subtask

Polynomial functions can be represented in a number of ways: algebraically, graphically, numerically in tables, or by verbal descriptions. Properties, theorems, and technological tools allow you to analyze and compare polynomial functions regardless of the way in which they are represented.

Each of the representations below is a representation of a fourth-degree polynomial.

A. $f(x) = -2x^4 + 10x^2 + 72$ **B.**

1. Work with your group. Use any method you like to answer the following questions. Justify each answer. As you justify your answers, speak clearly and use precise mathematical language to describe your reasoning and your conclusions. Remember to use complete sentences, including transitions and words such as *and, or, since, for example, therefore, because of* to make connections between your thoughts.

    a. Which polynomial has the larger maximum value?

    b. Which polynomial has more real roots?

> ### Check Your Understanding
>
> 2. **Construct viable arguments.** Which of the polynomials above has the larger *y*-intercept? Justify your answer.
>
> 3. For which of the polynomials above is $f(1000)$ smaller? Justify your answer.

My Notes

MATH TIP

The portions of a graph that are below the x-axis satisfy the inequality $f(x) < 0$, while the portions of the graph that are above the x-axis satisfy the inequality $f(x) > 0$.

To solve a **polynomial inequality** by graphing, use the fact that a polynomial can only change signs at its zeros.

Step 1: Write the polynomial inequality with one side equal to zero.

Step 2: Graph the inequality and determine the zeros.

Step 3: Find the intervals where the conditions of the inequality are met.

4. **Use appropriate tools strategically.** Solve the polynomial inequality $x^4 - 13x^2 + 6 < -30$ by graphing on a graphing calculator or by hand.

Check Your Understanding

5. Solve the polynomial inequality $(x + 9)(x + 2)(x - 4) > 0$ by graphing.

6. Solve the polynomial inequality $x^3 - 2x < 0$ by graphing.

LESSON 18-3 PRACTICE

7. Which representation below is a quadratic function that has zeros at $x = -4$ and $x = 2$? Justify your answer.

   A. $h(x) = x^2 + 2x - 8$

   B.

x	0	5	4	0	-2
y	-4	-1	0	2	0

8. **Make sense of problems.** The function $f(x)$ is a polynomial that decreases without bound as $x \to \pm\infty$, has a double root at $x = 0$, and has no other real roots. The function $g(x)$ is given by the equation $g(x) = -x^4 + 16$. Which function has the greater range? Explain your reasoning.

9. The graph of $q(x)$ is shown below. Use the graph to solve $q(x) \geq 0$.
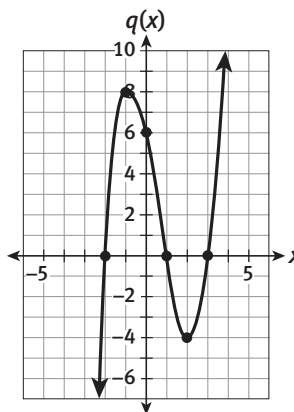
10. Solve the polynomial inequality $x^4 - 26x^2 < -25$.

11. **Reason abstractly.** Give an example of a quadratic function for which $f(x) > 0$ is true for all real numbers. Explain your reasoning.

ACTIVITY 18 PRACTICE

Write your answers on notebook paper.
**Show your work.**

Lesson 18-1

For Items 1–8, match each equation or description to one of the graphs below.

_____ **1.** an even function with no real roots and a positive leading coefficient

_____ **2.** an even function with three real roots and a negative leading coefficient.

_____ **3.** an odd function with one real root and a negative leading coefficient.

_____ **4.** $f(x) = -ax^3 + b$

_____ **5.** $g(x) = ax^3 + \ldots + d$

_____ **6.** $h(x) = ax^4 + \ldots - e$

_____ **7.** $p(x) = ax^5 + \ldots - f$

_____ **8.** $p(x) = -ax^5 + \ldots - f$

A.

B.

C.

D.

E.

F.

G.

H.
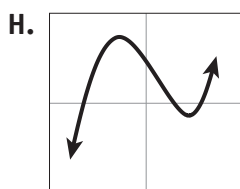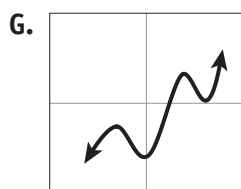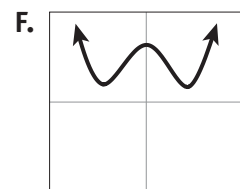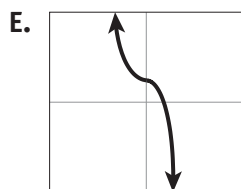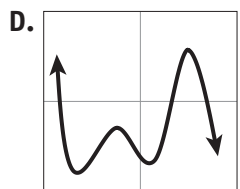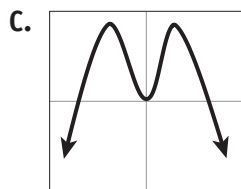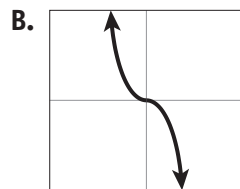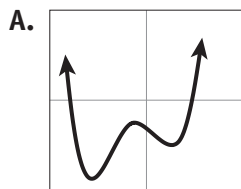
For Items 9–11, use what you know about end behavior and zeros to graph each function.

9. $f(x) = x^4 + 2x^3 - 43x^2 - 44x + 84$
$= (x - 1)(x - 6)(x + 2)(x + 7)$

10. $y = x^5 - 14x^4 + 37x^3 + 260x^2 - 1552x + 2240$
$= (x - 7)(x + 5)(x - 4)^3$

11. $f(x) = -x^4 + 11x^3 - 21x^2 - 59x + 70$
$= -(x - 1)(x - 5)(x + 2)(x - 7)$

12. Make a general statement about what information is revealed by an unfactored polynomial compared to a factored polynomial.

13. Miguel identified the graph below as a polynomial function of the form $f(x) = ax^4 - bx^2 + c$, where a, b, and c are positive real numbers.
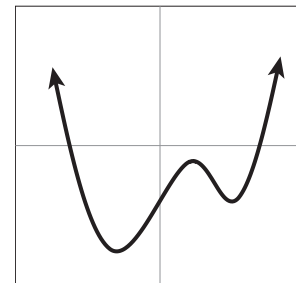
Which reason best describes why Miguel is incorrect?
  A. The graph is not a fourth-degree polynomial.
  B. The leading coefficient of Miguel's polynomial should be negative.
  C. The graph is of an even function, but Miguel's polynomial is not even.
  D. The y-intercept is below the x-axis, so Miguel's polynomial should end with $-c$, not $+c$.

Lesson 18-2

14. Determine all the possible rational roots of:
  a. $f(x) = -4x^3 - 13x^2 - 6x - 3$
  b. $g(x) = 2x^4 + 6x^3 - 3x^2 - 11x + 8$

15. Graph $f(x) = -4x^3 - 13x^2 - 6x - 3$.

16. Determine the possible number of positive and negative real roots for:
  a. $h(x) = 2x^3 + x^2 - 5x + 2$
  b. $p(x) = 2x^4 + 6x^3 - 3x^2 - 11x + 8$

17. Graph $h(x) = 2x^3 + x^2 - 5x + 2$.

18. Descartes' Rule of Signs states that the number of positive real roots of $f(x) = 0$ equals the number of variations in sign of the terms of $f(x)$, or is less than this number by an even integer. What theorem offers a reason as to why the number could be "less than this number by an <u>even</u> integer"?

For Items 19–20, apply the Remainder Theorem to all the possible rational roots of the given polynomial to identify points on the graph or zeros of the polynomial.

19. $p(x) = x^3 - 5x^2 + 8x - 4$

20. $h(x) = 2x^4 + 5x^3 - x^2 + 5x - 3$

21. The graph of $f(x)$ has an x-intercept at $(4, 0)$. Which of the following MUST be true?

  I. $f(4) = 0$
  II. $x - 4$ is a factor of $f(x)$.
  III. $f(x)$ also has an x-intercept at $(-4, 0)$.

  A. II only
  B. I and II only
  C. II and III only
  D. I, II, and III

Lesson 18-3

For Items 22–24, solve the polynomial inequality.

22. $(x + 4)(x - 2)(x - 10) > 0$

23. $x^3 - x^2 - 36x + 36 < 0$

24. $-x^4 + 20x^2 - 32 \geq 32$

MATHEMATICAL PRACTICES
Look For and Express Regularity in Repeated Reasoning

25. Some polynomial functions are represented in a variety of forms below. For each representation, describe whether you think it is more efficient to graph the polynomial using a graphing calculator or by hand. Justify your choices.

  a. $f(x) = (x + 15)(x + 7)(x - 5)^2(x - 12)$

  b. $g(x) = 2x^4 + 6x^3 - 3x^2 - 11x + 7$

  c.

| x	$f(x)$
-3	-8
-1	1
0	2
1	1
3	-6
4	-2

1. Factor $f(x) = x^3 + 3x^2 - x - 3$. Then find the zeros and y-intercept. Sketch a graph of the function.

2. Find two different ways to show that $g(x) = -x^3 + 27$ has only one x-intercept. Use a sketch of the graph as one method, if necessary.

3. List all the characteristics of the graph for this polynomial function that you would expect to see, based on what you have learned thus far.

$$f(x) = (x + 3)(x - 3)(x + 2)(x - 1)(x + 2i)(x - 2i)$$

4. Find a polynomial function of fourth degree that has the zeros 2, −2, and $1 - 3i$. Then write it in standard form.

5. The graph below represents a fourth-degree polynomial function with no imaginary roots.
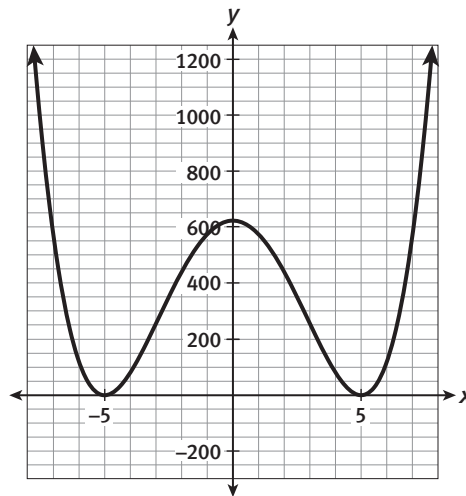
   a. Is the function even, odd, or neither? Explain your reasoning.
   b. State the domain and range of the function.
   c. Given that $f(-5) = 0$ and $f(5) = 0$, use the graph to find the equation of the function, in factored form and in standard form.
   d. Explain how you can use the y-intercept shown on the graph to check that your equation is correct.

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
	The solution demonstrates these characteristics:			
Mathematics Knowledge and Thinking (Items 1-5)	• Clear and accurate understanding of how to rewrite polynomials in equivalent forms • Effective understanding and identification of the features of a polynomial function and its graph, including even and odd functions • Effective understanding of the relationship between the factors and zeros of a polynomial function, including complex zeros	• Largely correct understanding of how to rewrite polynomials in equivalent forms • A functional understanding and accurate identification of the features of a polynomial function and its graph, including even and odd functions • A functional understanding of the relationship between the factors and zeros of a polynomial function, including complex zeros	• Difficulty when rewriting polynomials in equivalent forms • Partial understanding and partially accurate identification of the features of a polynomial function and its graph, including even and odd functions • Partial understanding of the relationship between the factors and zeros of a polynomial function, including complex zeros	• Inaccurate or incomplete understanding of how to rewrite polynomials in equivalent forms • Little or no understanding and inaccurate identification of the features of a polynomial function and its graph, including even and odd functions • Little or no understanding of the relationship between the factors and zeros of a polynomial function, including complex zeros
Problem Solving (Items 2, 5d)	• An appropriate and efficient strategy that results in a correct answer	• A strategy that may include unnecessary steps but results in a correct answer	• A strategy that results in some incorrect answers	• No clear strategy when solving problems
Mathematical Modeling / Representations (Items 1, 4, 5c)	• Fluency in sketching the graph of a polynomial function, given the equation in factored form • Fluency in finding the equation for a polynomial function, given the roots or a graph	• Little difficulty in sketching the graph of a polynomial function, given the equation in factored form • Little difficulty in finding the equation for a polynomial function, given the roots or a graph	• Partial understanding of sketching the graph of a polynomial function, given the equation in factored form • Partial understanding of finding the equation for a polynomial function, given the roots or a graph	• Little or no understanding of sketching the graph of a polynomial function, given the equation in factored form • Little or no understanding of finding the equation for a polynomial function, given the roots or a graph
Reasoning and Communication (Items 2, 3, 5a, 5d)	• Precise use of appropriate math terms and language to describe the features of the graph of a polynomial function • Clear and accurate explanations of why a function has one intercept and whether a function is even, odd, or neither • Clear and accurate explanation of how to use the *y*-intercept to check an equation for a graph	• Adequate description of the features of the graph of a polynomial function • Adequate explanation of why a function has one intercept and whether a function is even, odd, or neither • Adequate explanation of how to use the *y*-intercept to check an equation for a graph	• Misleading or confusing description of the features of the graph of a polynomial function • Misleading or confusing explanation of why a function has one intercept and whether a function is even, odd, or neither • Misleading or confusing explanation of how to use the *y*-intercept to check an equation for a graph	• Incomplete or inaccurate description of the features of the graph of a polynomial function • Incomplete or inadequate explanation of why a function has one intercept and whether a function is even, odd, or neither • Incomplete or inadequate explanation of how to use the *y*-intercept to check an equation for a graph