

# Quadratic Functions

2

## Unit Overview

This unit focuses on quadratic functions and equations. You will write the equations of quadratic functions to model situations. You will also graph quadratic functions and other parabolas and interpret key features of the graphs. In addition, you will study methods of finding solutions of quadratic equations and interpreting the meaning of the solutions. You will also extend your knowledge of number systems to the complex numbers.

## Key Terms

As you study this unit, add these and other terms to your math notebook. Include in your notes your prior knowledge of each word, as well as your experiences in using the word in different mathematical examples. If needed, ask for help in pronouncing new words and add information on pronunciation to your math notebook. It is important that you learn new terms and use them correctly in your class discussions and in your problem solutions.

## Academic Vocabulary

- justify
- derive
- verify
- advantage
- disadvantage
- counterexample

## Math Terms

- quadratic equation
- standard form of a quadratic equation
- imaginary number
- complex number
- complex conjugate
- completing the square
- discriminant
- root
- zero
- parabola
- focus
- directrix
- axis of symmetry
- vertex
- quadratic regression
- vertex form

## ESSENTIAL QUESTIONS

- ? How can you determine key attributes of a quadratic function from an equation or graph?
- ? How do graphic, symbolic, and numeric methods of solving quadratic equations compare to one another?

## EMBEDDED ASSESSMENTS

This unit has three embedded assessments, following Activities 9, 11, and 13. By completing these embedded assessments, you will demonstrate your understanding of key features of quadratic functions and parabolas, solutions to quadratic equations, and systems that include nonlinear equations.

### Embedded Assessment 1:

Applications of Quadratic Functions and Equations p. 151

### Embedded Assessment 2:

Writing and Transforming Quadratic Functions p. 191

### Embedded Assessment 3:

Graphing Quadratic Functions and Solving Systems p. 223

# Getting Ready

Write your answers on notebook paper.  
Show your work.

Factor the expressions in Items 1–4 completely.

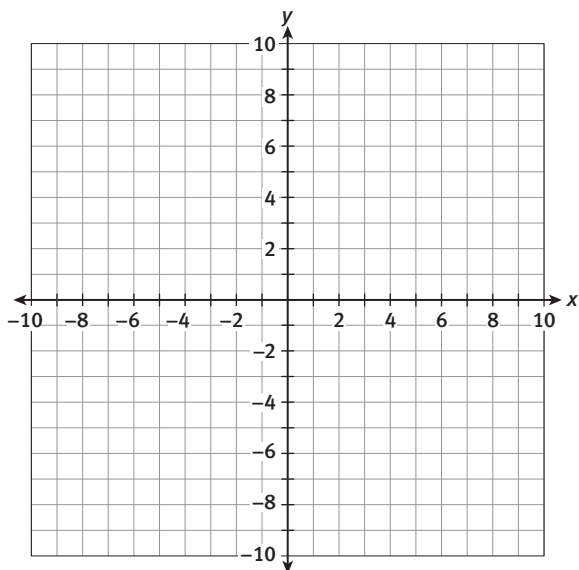
1.  $6x^3y + 12x^2y^2$

2.  $x^2 + 3x - 40$

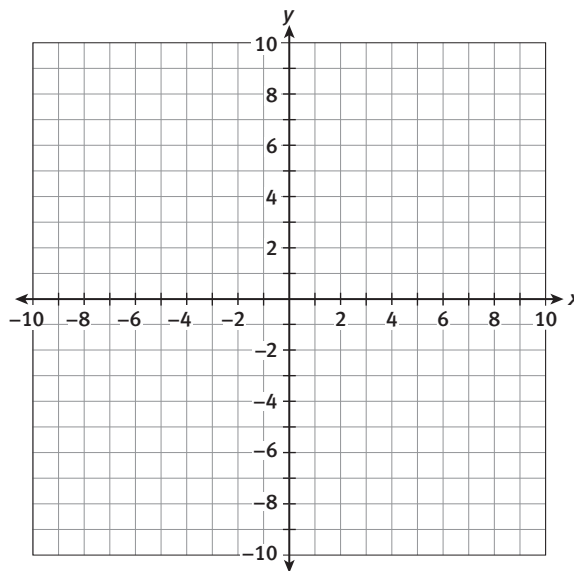
3.  $x^2 - 49$

4.  $x^2 - 6x + 9$

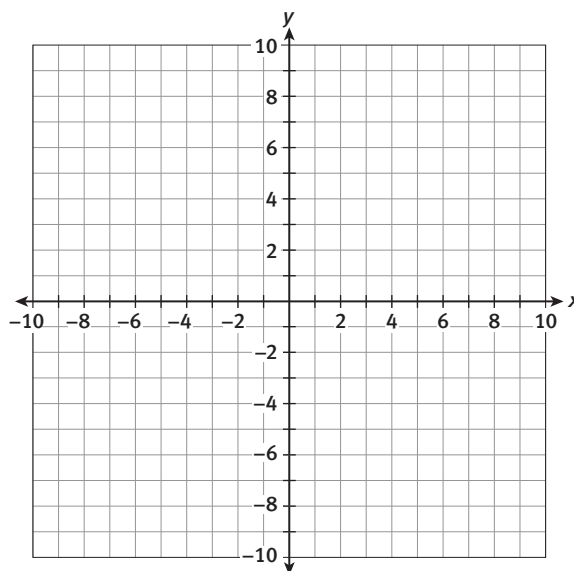
5. Graph  $f(x) = \frac{3}{4}x - \frac{3}{2}$ .



6. Graph a line that has an  $x$ -intercept of 5 and a  $y$ -intercept of  $-2$ .



7. Graph  $y = |x|$ ,  $y = |x + 3|$ , and  $y = |x| + 3$  on the same grid.



8. Solve  $x^2 - 3x - 5 = 0$ .

### Fences

#### Lesson 7-1 Analyzing a Quadratic Function

#### Learning Targets:

- Formulate quadratic functions in a problem-solving situation.
- Graph and interpret quadratic functions.

**SUGGESTED LEARNING STRATEGIES:** Marking the Text, Guess and Check, Create Representations, Quickwrite, Self Revision/Peer Revision

Fence Me In is a business that specializes in building fenced enclosures. One client has purchased 100 ft of fencing to enclose the largest possible rectangular area in her yard.

Work with your group on Items 1–7. As you share ideas, be sure to explain your thoughts using precise language and specific details to help group members understand your ideas and your reasoning.

1. If the width of the rectangular enclosure is 20 ft, what must be the length? Find the area of this rectangular enclosure.
2. Choose several values for the width of a rectangle with a perimeter of 100 ft. Determine the corresponding length and area of each rectangle. Share your values with members of your class. Then record each set of values in the table below.

Width (ft)	Length (ft)	Area (ft <sup>2</sup> )

3. **Make sense of problems.** What is the relationship between the length and width of a rectangle with perimeter of 100 ft?
4. Based on your observations, predict if it is possible for a rectangle with perimeter of 100 ft to have each area. Explain your reasoning.
  - a. 400 ft<sup>2</sup>
  - b. 500 ft<sup>2</sup>

My Notes

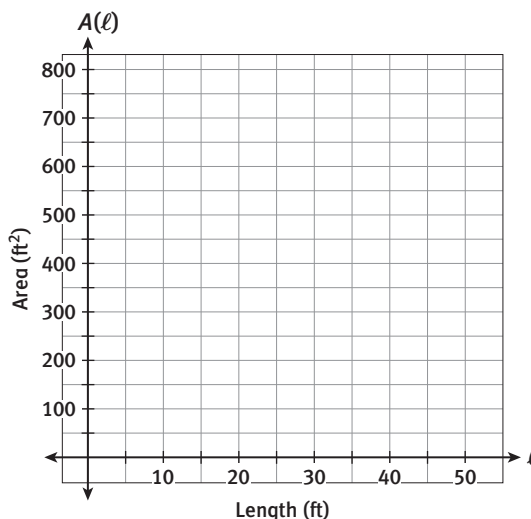
#### DISCUSSION GROUP TIP

Reread the problem scenario as needed. Make notes on the information provided in the problem. Respond to questions about the meaning of key information. Summarize or organize the information needed to create reasonable solutions, and describe the mathematical concepts your group will use to create its solutions.

My Notes

c.  $700 \text{ ft}^2$

5. Let  $l$  represent the length of a rectangle with a perimeter of 100 ft. Write an expression for the width of the rectangle in terms of  $l$ .
6. Express the area  $A(l)$  for a rectangle with a perimeter of 100 ft as a function of its length,  $l$ .
7. Graph the quadratic function  $A(l)$  on the coordinate grid.



TECHNOLOGY TIP

To graph the function on a graphing calculator, you will first need to substitute  $y$  for  $A(l)$  and  $x$  for  $l$  before you can enter the equation.

8. **Use appropriate tools strategically.** Now use a graphing calculator to graph the quadratic function  $A(l)$ . Set your window to correspond to the values on the axes on the graph in Item 7.
9. Use the function  $A(l)$  and your graphs from Items 7 and 8 to complete the following.
  - a. What is the reasonable domain of the function in this situation? Express the domain as an inequality, in interval notation, and in set notation.

## Lesson 7-1

### Analyzing a Quadratic Function

## ACTIVITY 7

continued

- b. Over what interval of the domain is the value of the function increasing? Over what interval of the domain is the value of the function decreasing?
10. What is the maximum rectangular area that can be enclosed by 100 ft of fencing? *Justify* your answer.
11. a. What is the reasonable range of  $A(\ell)$  in this situation? Express the range as an inequality, in interval notation, and in set notation.
- b. Explain how your answer to Item 10 helped you determine the reasonable range.
12. **Reason quantitatively.** Revise or confirm your predictions from Item 4. If a rectangle is possible, estimate its dimensions and explain your reasoning. Review the draft of your revised or confirmed predictions. Be sure to check that you have included specific details, the correct mathematical terms to support your explanations, and that your sentences are complete and grammatically correct. You may want to pair-share with another student to critique each other's drafts and make improvements.
- a.  $400 \text{ ft}^2$
- b.  $500 \text{ ft}^2$
- c.  $700 \text{ ft}^2$

My Notes

### ACADEMIC VOCABULARY

When you *justify* an answer, you show that your answer is correct or reasonable.

### CONNECT TO AP

The process of finding the maximum (or minimum) value of a function is called *optimization*, a topic addressed in calculus.

## My Notes

13. What are the length and width of the largest rectangular area that can be enclosed by 100 ft of fencing?
14. The length you gave in Item 13 is the solution of a quadratic equation in terms of  $l$ . Write this equation. Explain how you arrived at this equation.

## Check Your Understanding

15. Explain why the function  $A(l)$  that you used in this lesson is a quadratic function.
16. How does the graph of a quadratic function differ from the graph of a linear function?
17. Can the range of a quadratic function be all real numbers? Explain.
18. Explain how you could solve the quadratic equation  $x^2 + 2x = 3$  by graphing the function  $f(x) = x^2 + 2x$ .

## LESSON 7-1 PRACTICE

For Items 19–21, consider a rectangle that has a perimeter of 120 ft.

19. Write a function  $B(l)$  that represents the area of the rectangle with length  $l$ .
20. Graph the function  $B(l)$ , using a graphing calculator. Then copy it on your paper, labeling axes and using an appropriate scale.
21. Use the graph of  $B(l)$  to find the dimensions of the rectangle with a perimeter of 120 feet that has each area. Explain your answer.
  - a.  $500 \text{ ft}^2$
  - b.  $700 \text{ ft}^2$
22. **Critique the reasoning of others.** An area of  $1000 \text{ ft}^2$  is not possible. Explain why this is true.
23. How is the maximum value of a function shown on the graph of the function? How would a minimum value be shown?



**Learning Targets:**

- Factor quadratic expressions of the form  $x^2 + bx + c$ .
- Factor quadratic expressions of the form  $ax^2 + bx + c$ .

**SUGGESTED LEARNING STRATEGIES:** Interactive Word Wall, Vocabulary Organizer, Marking the Text, Guess and Check, Work Backward, RAFT

In the previous lesson, you used the function  $A(l) = -l^2 + 50l$  to model the area in square feet of a rectangle that can be enclosed with 100 ft of fencing.

1. **Reason quantitatively.** What are the dimensions of the rectangle if its area is 525 ft<sup>2</sup>? Explain how you determined your answer.

2. One way to find the dimensions of the rectangle is to solve a quadratic equation algebraically. What **quadratic equation** could you have solved to answer Item 1?

3. Write the quadratic equation from Item 2 in the form  $al^2 + bl + c = 0$ , where  $a > 0$ . Give the values of  $a$ ,  $b$ , and  $c$ .

As you have seen, graphing is one way to solve a quadratic equation. However, you can also solve quadratic equations algebraically by factoring.

You can use the graphic organizer shown in Example A on the next page to recall factoring trinomials of the form  $x^2 + bx + c = 0$ . Later in this activity, you will solve the quadratic equation from Item 3 by factoring.

My Notes

**MATH TERMS**

A **quadratic equation** can be written in the form  $ax^2 + bx + c = 0$ , where  $a \neq 0$ . An expression in the form  $ax^2 + bx + c$ ,  $a \neq 0$ , is a **quadratic expression**.

My Notes

**Example A**

Factor  $x^2 + 12x + 32$ .

**Step 1:** Place  $x^2$  in the upper left box and the constant term 32 in the lower right.

$x^2$	
	32

**Step 2:** List factor pairs of 32, the constant term. Choose the pair that has a sum equal to 12, the coefficient  $b$  of the  $x$ -term.

Factors		Sum
32	1	$32 + 1 = 33$
16	2	$16 + 2 = 18$
8	4	$8 + 4 = 12$

**Step 3:** Write each factor as coefficients of  $x$  and place them in the two empty boxes. Write common factors from each row to the left and common factors for each column above.

	$x$	8
$x$	$x^2$	$8x$
4	$4x$	32

**Step 4:** Write the sum of the common factors as binomials. Then write the factors as a product.

$(x + 4)(x + 8)$

**Solution:**  $x^2 + 12x + 32 = (x + 4)(x + 8)$

**Try These A**

a. Factor  $x^2 - 7x + 12$ , using the graphic organizer. Then check by multiplying.


Factor, and then check by multiplying. Show your work.

b.  $x^2 + 9x + 14$

c.  $x^2 - 7x - 30$

d.  $x^2 - 12x + 36$

e.  $x^2 - 144$

f.  $5x^2 + 40x + 75$

g.  $-12x^2 + 108$

**MATH TIP**

To check that your factoring is correct, multiply the two binomials by distributing.

$$\begin{aligned} &(x + 4)(x + 8) \\ &= x^2 + 4x + 8x + 32 \\ &= x^2 + 12x + 32 \end{aligned}$$

**MATH TIP**

A difference of squares  $a^2 - b^2$  is equal to  $(a - b)(a + b)$ . A perfect square trinomial  $a^2 + 2ab + b^2$  is equal to  $(a + b)^2$ .



## Lesson 7-2

### Factoring Quadratic Expressions

## ACTIVITY 7

continued

Before factoring quadratic expressions  $ax^2 + bx + c$ , where the leading coefficient  $a \neq 1$ , consider how multiplying binomial factors results in that form of a quadratic expression.

**4. Make sense of problems.** Use a graphic organizer to multiply  $(2x + 3)(4x + 5)$ .

a. Complete the graphic organizer by filling in the two empty boxes.

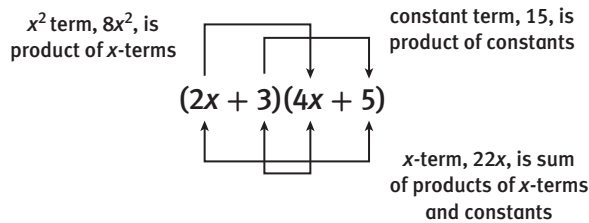
b.  $(2x + 3)(4x + 5)$

$$= 8x^2 + \underline{\hspace{2cm}} + \underline{\hspace{2cm}} + 15$$

$$= 8x^2 + \underline{\hspace{2cm}} + 15$$

	$2x$	$3$
$4x$	$8x^2$	
$5$		$15$

Using the Distributive Property, you can see the relationship between the numbers in the binomial factors and the terms of the trinomial.



To factor a quadratic expression  $ax^2 + bx + c$ , work backward from the coefficients of the terms.

### Example B

Factor  $6x^2 + 13x - 5$ . Use a table to organize your work.

**Step 1:** Identify the factors of 6, which is  $a$ , the coefficient of the  $x^2$ -term.

**Step 2:** Identify the factors of  $-5$ , which is  $c$ , the constant term.

**Step 3:** Find the numbers whose products add together to equal 13, which is  $b$ , the coefficient of the  $x$ -term.

**Step 4:** Then write the binomial factors.

Factors of 6	Factors of $-5$	Sum = 13?
1 and 6	$-1$ and $5$	$1(5) + 6(-1) = -1$
1 and 6	$5$ and $-1$	$1(-1) + 6(5) = 29$
2 and 3	$-1$ and $5$	$2(5) + 3(-1) = 7$
2 and 3	$5$ and $-1$	$2(-1) + 3(5) = 13$ ✓

**Solution:**  $6x^2 + 13x - 5 = (2x + 5)(3x - 1)$

### My Notes

### MATH TIP

Check your answer by multiplying the two binomials.

$$\begin{aligned} &(2x + 5)(3x - 1) \\ &= 6x^2 - 2x + 15x - 5 \\ &= 6x^2 + 13x - 5 \end{aligned}$$

My Notes

Try These B

Factor, and then check by multiplying. Show your work.

- a.  $10x^2 + 11x + 3$
- b.  $4x^2 + 17x - 15$
- c.  $2x^2 - 13x + 21$
- d.  $6x^2 - 19x - 36$

Check Your Understanding

- 5. Explain how the graphic organizer shows that  $x^2 + 8x + 15$  is equal to  $(x + 5)(x + 3)$ .
- 6. **Reason abstractly.** Given that  $b$  is negative and  $c$  is positive in the quadratic expression  $x^2 + bx + c$ , what can you conclude about the signs of the constant terms in the factored form of the expression? Explain your reasoning.
- 7. Write a set of instructions for a student who is absent, explaining how to factor the quadratic expression  $x^2 + 4x - 12$ .

	$x$	$5$
$x$	$x^2$	$5x$
$3$	$3x$	$15$

LESSON 7-2 PRACTICE

Factor each quadratic expression.

- 8.  $2x^2 + 15x + 28$
- 9.  $3x^2 + 25x - 18$
- 10.  $x^2 + x - 30$
- 11.  $x^2 + 15x + 56$
- 12.  $6x^2 - 7x - 5$
- 13.  $12x^2 - 43x + 10$
- 14.  $2x^2 + 5x$
- 15.  $9x^2 - 3x - 2$
- 16. A customer of Fence Me In wants to increase both the length and width of a rectangular fenced area in her backyard by  $x$  feet. The new area in square feet enclosed by the fence is given by the expression  $x^2 + 30x + 200$ .
  - a. Factor the quadratic expression.
  - b. **Reason quantitatively.** What were the original length and width of the fenced area? Explain your answer.

**Learning Targets:**

- Solve quadratic equations by factoring.
- Interpret solutions of a quadratic equation.
- Create quadratic equations from solutions.

**SUGGESTED LEARNING STRATEGIES:** Marking the Text, Paraphrasing, Think-Pair-Share, Create Representations, Quickwrite

To solve a quadratic equation  $ax^2 + bx + c = 0$  by factoring, the equation must be in factored form to use the Zero Product Property.

**Example A**

Solve  $x^2 + 5x - 14 = 0$  by factoring.  
Original equation

$$x^2 + 5x - 14 = 0$$

**Step 1:** Factor the left side.

$$(x + 7)(x - 2) = 0$$

**Step 2:** Apply the Zero Product Property.  $x + 7 = 0$  or  $x - 2 = 0$

**Step 3:** Solve each equation for  $x$ .

**Solution:**  $x = -7$  or  $x = 2$

**Try These A**

a. Solve  $3x^2 - 17x + 10 = 0$  and check by substitution.

	Original equation
	Factor the left side.
	Apply the Zero Product Property.
	Solve each equation for $x$ .

Solve each equation by factoring. Show your work.

b.  $12x^2 - 7x - 10 = 0$     c.  $x^2 + 8x - 9 = 0$     d.  $4x^2 + 12x + 9 = 0$

e.  $18x^2 - 98 = 0$     f.  $x^2 + 6x = -8$     g.  $5x^2 + 2x = 3$

My Notes

**MATH TIP**

The *Zero Product Property* states that if  $a \cdot b = 0$ , then either  $a = 0$  or  $b = 0$ .

**MATH TIP**

You can check your solutions by substituting the values into the original equation.

## My Notes

In the previous lesson, you were asked to determine the dimensions of a rectangle with an area of  $525 \text{ ft}^2$  that can be enclosed by 100 ft of fencing. You wrote the quadratic equation  $l^2 - 50l + 525 = 0$  to model this situation, where  $l$  is the length of the rectangle in feet.

1.
  - a. Solve the quadratic equation by factoring.
  - b. What do the solutions of the equation represent in this situation?
  - c. What are the dimensions of a rectangle with an area of  $525 \text{ ft}^2$  that can be enclosed by 100 ft of fencing?
  - d. **Reason quantitatively.** Explain why your answer to part c is reasonable.
2. A park has two rectangular tennis courts side by side. Combined, the courts have a perimeter of 160 yd and an area of  $1600 \text{ yd}^2$ .
  - a. Write a quadratic equation that can be used to find  $l$ , the length of the court in yards.
  - b. **Construct viable arguments.** Explain why you need to write the equation in the form  $al^2 + bl + c = 0$  before you can solve it by factoring.
  - c. Solve the quadratic equation by factoring, and interpret the solution.
  - d. Explain why the quadratic equation has only one distinct solution.

## MATH TIP

It is often easier to factor a quadratic equation if the coefficient of the  $x^2$ -term is positive. If necessary, you can multiply both sides of the equation by  $-1$  to make the coefficient positive.

## Lesson 7-3

### Solving Quadratic Equations by Factoring

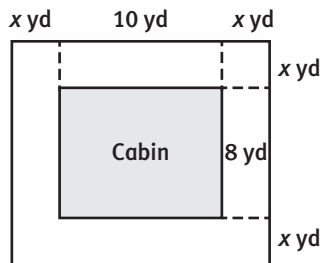
## ACTIVITY 7

continued

3. The equation  $2x^2 + 9x - 3 = 0$  cannot be solved by factoring. Explain why this is true.

### Check Your Understanding

4. Explain how to use factoring to solve the equation  $2x^2 + 5x = 3$ .
5. **Critique the reasoning of others.** A student incorrectly states that the solution of the equation  $x^2 + 2x - 35 = 0$  is  $x = -5$  or  $x = 7$ . Describe the student's error, and solve the equation correctly.
6. Fence Me In has been asked to install a fence around a cabin. The cabin has a length of 10 yd and a width of 8 yd. There will be a space  $x$  yd wide between the cabin and the fence on all sides, as shown in the diagram. The area to be enclosed by the fence is  $224 \text{ yd}^2$ .
- a. Write a quadratic equation that can be used to determine the value of  $x$ .
- b. Solve the equation by factoring.
- c. Interpret the solutions.



If you know the solutions to a quadratic equation, then you can write the equation.

### Example B

Write a quadratic equation in **standard form** with the solutions  $x = 4$  and  $x = -5$ .

- Step 1:** Write linear equations that correspond to the solutions.  $x - 4 = 0$  or  $x + 5 = 0$
- Step 2:** Write the linear expressions as factors.  $(x - 4)$  and  $(x + 5)$
- Step 3:** Multiply the factors to write the equation in factored form.  $(x - 4)(x + 5) = 0$
- Step 4:** Multiply the binomials and write the equation in standard form.  $x^2 + x - 20 = 0$

**Solution:**  $x^2 + x - 20 = 0$  is a quadratic equation with solutions  $x = 4$  and  $x = -5$ .

### MATH TERMS

The **standard form of a quadratic equation** is  $ax^2 + bx + c = 0$ , where  $a \neq 0$ .

My Notes

Try These B

- a. Write a quadratic equation in standard form with the solutions  $x = -1$  and  $x = -7$ .

	Write linear equations that correspond to the solutions.
	Write the linear expressions as factors.
	Multiply the factors to write the equation in factored form.
	Multiply the binomials and write the equation in standard form.

- b. Write a quadratic equation in standard form whose solutions are  $x = \frac{2}{5}$  and  $x = -\frac{1}{2}$ . How is your result different from those in Example B?

MATH TIP

To avoid fractions as coefficients, multiply the coefficients by the LCD.

Write a quadratic equation in standard form with integer coefficients for each pair of solutions. Show your work.

c.  $x = \frac{2}{3}, x = 2$

d.  $x = -\frac{3}{2}, x = \frac{5}{2}$

Check Your Understanding

- Write the equation  $3x^2 - 6x = 10x + 12$  in standard form.
- Explain how you could write the equation  $x^2 - \frac{7}{6}x + \frac{1}{3} = 0$  with integer values of the coefficients and constants.
- Reason quantitatively.** Is there more than one quadratic equation whose solutions are  $x = -3$  and  $x = -1$ ? Explain.
- How could you write a quadratic equation in standard form whose only solution is  $x = 4$ ?



## Lesson 7-3

### Solving Quadratic Equations by Factoring

## ACTIVITY 7

continued

### LESSON 7-3 PRACTICE

Solve each quadratic equation by factoring.

11.  $2x^2 - 11x + 5 = 0$

12.  $x^2 + 2x = 15$

13.  $3x^2 + x - 4 = 0$

14.  $6x^2 - 13x - 5 = 0$

Write a quadratic equation in standard form with integer coefficients for which the given numbers are solutions.

15.  $x = 2$  and  $x = -5$

16.  $x = -\frac{2}{3}$  and  $x = -5$

17.  $x = \frac{3}{5}$  and  $x = 3$

18.  $x = -\frac{1}{2}$  and  $x = \frac{3}{4}$

19. **Model with mathematics.** The manager of Fence Me In is trying to determine the best selling price for a particular type of gate latch. The function  $p(s) = -4s^2 + 400s - 8400$  models the yearly profit the company will make from the latches when the selling price is  $s$  dollars.
- Write a quadratic equation that can be used to determine the selling price that would result in a yearly profit of \$1600.
  - Write the quadratic equation in standard form so that the coefficient of  $s^2$  is 1.
  - Solve the quadratic equation by factoring, and interpret the solution(s).
  - Explain how you could check your answer to part c.

My Notes

#### CONNECT TO ECONOMICS

The selling price of an item has an effect on how many of the items are sold. The number of items that are sold, in turn, has an effect on the amount of profit a company makes by selling the item.

**My Notes**

**Learning Targets:**

- Solve quadratic inequalities.
- Graph the solutions to quadratic inequalities.

**SUGGESTED LEARNING STRATEGIES:** Identify a Subtask, Guess and Check, Think Aloud, Create Representations, Quickwrite

Factoring is also used to solve quadratic inequalities.

**MATH TIP**

For a product of two numbers to be positive, both factors must have the same sign. If the product is negative, then the factors must have opposite signs.

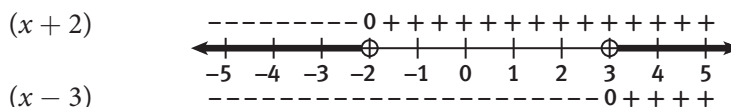
**Example A**

Solve  $x^2 - x - 6 > 0$ .

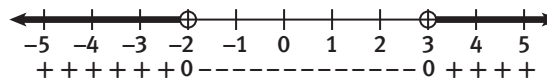
**Step 1:** Factor the quadratic expression on the left  $(x + 2)(x - 3) > 0$  side.

**Step 2:** Determine where each factor equals zero.  $(x + 2) = 0$  at  $x = -2$   
 $(x - 3) = 0$  at  $x = 3$

**Step 3:** Use a number line to visualize the intervals for which each factor is positive and negative. (Test a value in each interval to determine the signs.)



**Step 4:** Identify the sign of the product of the two factors on each interval.  $(x + 2)(x - 3)$

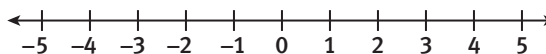


**Step 5:** Choose the appropriate interval. Since  $x^2 - x - 6$  is positive ( $> 0$ ), the intervals that show  $(x + 2)(x - 3)$  as positive represent the solutions.

**Solution:**  $x < -2$  or  $x > 3$

**Try These A**

a. Use the number line provided to solve  $2x^2 + x - 10 \leq 0$ .



Solve each quadratic inequality.

b.  $x^2 + 3x - 4 < 0$

c.  $3x^2 + x - 10 \geq 0$

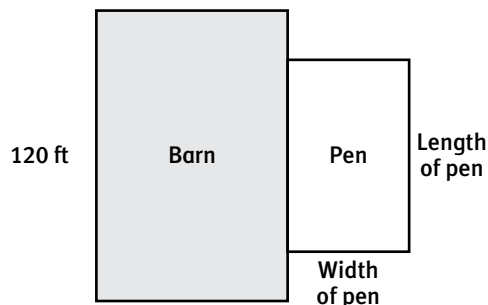
## Lesson 7-4

### More Uses for Factors

## ACTIVITY 7

continued

A farmer wants to enclose a rectangular pen next to his barn. A wall of the barn will form one side of the pen, and the other three sides will be fenced. He has purchased 100 ft of fencing and has hired Fence Me In to install it so that it encloses an area of at least  $1200 \text{ ft}^2$ .



Work with your group on Items 1–5. As you share ideas with your group, be sure to explain your thoughts using precise language and specific details to help group members understand your ideas and your reasoning.

- 1. Attend to precision.** If Fence Me In makes the pen 50 ft in length, what will be the width of the pen? What will be its area? Explain your answers.
2. Let  $l$  represent the length in feet of the pen. Write an expression for the width of the pen in terms of  $l$ .
3. Write an inequality in terms of  $l$  that represents the possible area of the pen. Explain what each part of your inequality represents.
4. Write the inequality in standard form with integer coefficients.
5. Use factoring to solve the quadratic inequality.

### My Notes

### DISCUSSION GROUP TIP

Reread the problem scenario as needed. Make notes on the information provided in the problem. Respond to questions about the meaning of key information. Summarize or organize the information needed to create reasonable solutions, and describe the mathematical concepts your group will use to create solutions.

### MATH TIP

If you multiply or divide both sides of an inequality by a negative number, you must reverse the inequality symbol.

## My Notes

6. Interpret the solutions of the inequality.
7. Use the possible lengths of the pen to determine the possible widths.

## Check Your Understanding

8. Consider the inequality  $(x + 4)(x - 5) \geq 0$ .
- Explain how to determine the intervals on a number line for which each of the factors  $(x + 4)$  and  $(x - 5)$  is positive or negative.
  - Reason abstractly.** How do you determine the sign of the product  $(x + 4)(x - 5)$  on each interval?
  - Once you know the sign of the product  $(x + 4)(x - 5)$  on each interval, how do you identify the solutions of the inequality?
9. Explain how the solutions of  $x^2 + 5x - 24 = 0$  differ from the solutions of  $x^2 + 5x - 24 \leq 0$ .
10. Explain why the quadratic inequality  $x^2 + 4 < 0$  has no real solutions.

## LESSON 7-4 PRACTICE

Solve each inequality.

11.  $x^2 + 3x - 10 \geq 0$
12.  $2x^2 + 3x - 9 < 0$
13.  $x^2 + 9x + 18 \leq 0$
14.  $3x^2 - 10x - 8 > 0$
15.  $x^2 - 12x + 27 < 0$
16.  $5x^2 + 12x + 4 > 0$
17. The function  $p(s) = -500s^2 + 15,000s - 100,000$  models the yearly profit Fence Me In will make from installing wooden fences when the installation price is  $s$  dollars per foot.
- Write a quadratic inequality that can be used to determine the installation prices that will result in a yearly profit of at least \$8000.
  - Write the quadratic inequality in standard form so that the coefficient of  $s^2$  is 1.
  - Make sense of problems.** Solve the quadratic inequality by factoring, and interpret the solution(s).

## ACTIVITY 7 PRACTICE

Write your answers on notebook paper.  
Show your work.

## Lesson 7-1

A rectangle has perimeter 40 cm. Use this information for Items 1–7.

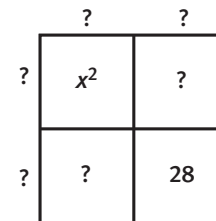
- Write the dimensions and areas of three rectangles that fit this description.
- Let the length of one side be  $x$ . Then write a function  $A(x)$  that represents that area of the rectangle.
- Graph the function  $A(x)$  on a graphing calculator. Then sketch the graph on grid paper, labeling the axes and using an appropriate scale.
- An area of  $96 \text{ cm}^2$  is possible. Use  $A(x)$  to demonstrate this fact algebraically and graphically.
- An area of  $120 \text{ cm}^2$  is not possible. Use  $A(x)$  to demonstrate this fact algebraically and graphically.
- What are the reasonable domain and reasonable range of  $A(x)$ ? Express your answers as inequalities, in interval notation, and in set notation.
- What is the greatest area that the rectangle could have? Explain.

Use the quadratic function  $f(x) = x^2 - 6x + 8$  for Items 8–11.

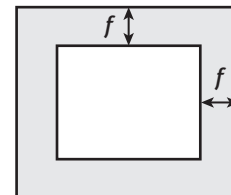
- Graph the function.
- Write the domain and range of the function as inequalities, in interval notation, and in set notation.
- What is the function's  $y$ -intercept?  
A. 0                      B. 2  
C. 4                      D. 8
- Explain how you could use the graph of the function to solve the equation  $x^2 - 6x + 8 = 3$ .

## Lesson 7-2

12. Factor  $x^2 + 11x + 28$  by copying and completing the graphic organizer. Then check by multiplying.



13. Factor each quadratic expression.
- $2x^2 - 3x - 27$
  - $4x^2 - 121$
  - $6x^2 + 11x - 10$
  - $3x^2 + 7x + 4$
  - $5x^2 - 42x - 27$
  - $4x^2 - 4x - 35$
  - $36x^2 - 100$
  - $12x^2 + 60x + 75$
14. Given that  $b$  is positive and  $c$  is negative in the quadratic expression  $x^2 + bx + c$ , what can you conclude about the signs of the constant terms in the factored form of the expression? Explain your reasoning.
15. The area in square inches of a framed photograph is given by the expression  $4f^2 + 32f + 63$ , where  $f$  is the width in inches of the frame.



- Factor the quadratic expression.
- What are the dimensions of the opening in the frame? Explain your answer.
- If the frame is 2 inches wide, what are the overall dimensions of the framed photograph? Explain your answer.

**Lesson 7-3**

16. Solve each quadratic equation by factoring.
- $2x^2 - 5x - 12 = 0$
  - $3x^2 + 7x = -2$
  - $4x^2 - 20x + 25 = 0$
  - $27x^2 - 12 = 0$
  - $6x^2 - 4 = 5x$
17. For each set of solutions, write a quadratic equation in standard form.
- $x = 5, x = -8$
  - $x = \frac{2}{3}, x = 4$
  - $x = -\frac{7}{5}, x = \frac{1}{2}$
  - $x = 6$
18. A student claims that you can find the solutions of  $(x - 2)(x - 3) = 2$  by solving the equations  $x - 2 = 2$  and  $x - 3 = 2$ . Is the student's reasoning correct? Explain why or why not.

One face of a building is shaped like a right triangle with an area of  $2700 \text{ ft}^2$ . The height of the triangle is 30 ft greater than its base. Use this information for Items 19–21.

19. Which equation can be used to determine the base  $b$  of the triangle in feet?
- $b(b + 30) = 2700$
  - $\frac{1}{2}b(b + 30) = 2700$
  - $b(b - 30) = 2700$
  - $\frac{1}{2}b(b - 30) = 2700$
20. Write the quadratic equation in standard form so that the coefficient of  $b^2$  is 1.
21. Solve the quadratic equation by factoring, and interpret the solutions. If any solutions need to be excluded, explain why.

**Lesson 7-4**

22. For what values of  $x$  is the product  $(x + 4)(x - 6)$  positive? Explain.
23. Solve each quadratic inequality.
- $x^2 - 3x - 4 \leq 0$
  - $3x^2 - 7x - 6 > 0$
  - $x^2 - 16x + 64 < 0$
  - $2x^2 + 8x + 6 \geq 0$
  - $x^2 - 4x - 21 \leq 0$
  - $5x^2 - 13x - 6 < 0$

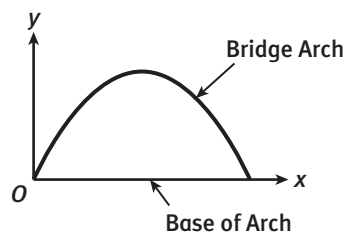
The function  $h(t) = -16t^2 + 20t + 6$  models the height in feet of a football  $t$  seconds after it is thrown. Use this information for Items 24–26.

24. Write a quadratic inequality that can be used to determine when the football will be at least 10 ft above the ground.
25. Write the quadratic inequality in standard form.
26. Solve the quadratic inequality by factoring, and interpret the solution(s).

**MATHEMATICAL PRACTICES**

**Make Sense of Problems and Persevere in Solving Them**

27. The graph of the function  $y = -\frac{1}{8}x^2 + 2x$  models the shape of an arch that forms part of a bridge, where  $x$  and  $y$  are the horizontal and vertical distances in feet from the left end of the arch.



- The greatest width of the arch occurs at its base. Use a graph to determine the greatest width of the arch. Explain how you used the graph to find the answer.
- Now write a quadratic equation that can help you find the greatest width of the arch. Solve the equation by factoring, and explain how you used the solutions to find the greatest width.
- Compare and contrast the methods of using a graph and factoring an equation to solve this problem.





## My Notes

## MATH TIP

You can solve a quadratic equation by graphing, by factoring, or by using the Quadratic Formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

You can use it to solve quadratic equations in the form

$$ax^2 + bx + c = 0, \text{ where } a \neq 0.$$

## CONNECT TO HISTORY

When considering his solutions, Cardano dismissed “mental tortures” and ignored the fact that  $\sqrt{x} \cdot \sqrt{x} = x$  only when  $x \geq 0$ .

## MATH TERMS

An **imaginary number** is any number of the form  $bi$ , where  $b$  is a real number and  $i = \sqrt{-1}$ .

- Solve your equation in Item 3 in two different ways. Explain each method.
- Write an equation that represents the problem that Cardano posed.
- Cardano claimed that the solutions to the problem are  $x = 5 + \sqrt{-15}$  and  $x = 5 - \sqrt{-15}$ . Verify his solutions by using the Quadratic Formula with the equation in Item 5.

Cardano avoided any more problems in *Ars Magna* involving the square root of a negative number. However, he did demonstrate an understanding about the properties of such numbers. Solving the equation  $x^2 + 1 = 0$  yields the solutions  $x = \sqrt{-1}$  and  $x = -\sqrt{-1}$ . The number  $\sqrt{-1}$  is represented by the symbol  $i$ , the imaginary unit. You can say  $i = \sqrt{-1}$ . The imaginary unit  $i$  is considered the solution to the equation  $x^2 + 1 = 0$ , or  $x^2 = -1$ .

To simplify an **imaginary number**  $\sqrt{-s}$ , where  $s$  is a positive number, you can write  $\sqrt{-s} = i\sqrt{s}$ .



My Notes

MATH TERMS

A **complex number** is a number in the form  $a + bi$ , where  $a$  and  $b$  are real numbers and  $i = \sqrt{-1}$ .

The set of complex numbers consists of the real numbers and the imaginary numbers. A **complex number** has two parts: the real part  $a$  and the imaginary part  $bi$ . For example, in  $2 + 3i$ , the real part is 2 and the imaginary part is  $3i$ .

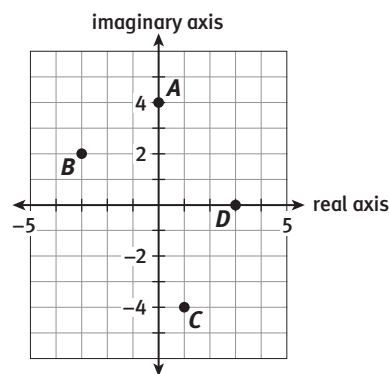
Check Your Understanding

12. Identify the real part and the imaginary part of each complex number.
  - a.  $5 + 8i$
  - b. 8
  - c.  $i\sqrt{10}$
  - d.  $\frac{5 + 3i}{2}$
13. Using the definition of complex numbers, show that the set of real numbers is a subset of the complex numbers.
14. Using the definition of complex numbers, show that the set of imaginary numbers is a subset of the complex numbers.

Complex numbers in the form  $a + bi$  can be represented geometrically as points in the *complex plane*. The complex plane is a rectangular grid, similar to the Cartesian plane, with the horizontal axis representing the real part  $a$  of a complex number and the vertical axis representing the imaginary part  $bi$  of a complex number. The point  $(a, b)$  on the complex plane represents the complex number  $a + bi$ .

Example B

- Point A represents  $0 + 4i$ .
- Point B represents  $-3 + 2i$ .
- Point C represents  $1 - 4i$ .
- Point D represents  $3 + 0i$ .



Try These B

- a. Graph  $2 + 3i$  and  $-3 - 4i$  on the complex plane above.
- Graph each complex number on a complex plane grid.
- b.  $2 + 5i$
  - c.  $4 - 3i$
  - d.  $-1 + 3i$
  - e.  $-2i$
  - f.  $-5$

## Lesson 8-1

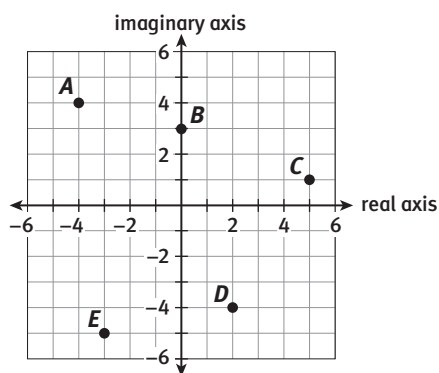
### The Imaginary Unit, $i$

## ACTIVITY 8

continued

### Check Your Understanding

- Reason abstractly.** Compare and contrast the Cartesian plane with the complex plane.
- What set of numbers do the points on the real axis of the complex plane represent? Explain.
- Name the complex number represented by each labeled point on the complex plane below.



### LESSON 8-1 PRACTICE

- Write each expression in terms of  $i$ .
  - $\sqrt{-49}$
  - $\sqrt{-13}$
  - $3 + \sqrt{-8}$
  - $5 - \sqrt{-36}$
- Identify the real part and the imaginary part of the complex number  $16 - i\sqrt{6}$ .
- Reason quantitatively.** Is  $\pi$  a complex number? Explain.
- Draw the complex plane. Then graph each complex number on the plane.
  - $6i$
  - $3 + 4i$
  - $-2 - 5i$
  - $4 - i$
  - $-3 + 2i$
- The sum of two numbers is 8, and their product is 80.
  - Let  $x$  represent one of the numbers, and write an expression for the other number in terms of  $x$ . Use the expressions to write an equation that models the situation given above.
  - Use the Quadratic Formula to solve the equation. Write the solutions in terms of  $i$ .

My Notes

### MATH TIP

$\pi$  is the ratio of a circle's circumference to its diameter.  $\pi$  is an irrational number, and its decimal form neither terminates nor repeats.

My Notes

**Learning Targets:**

- Add and subtract complex numbers.
- Multiply and divide complex numbers.

**SUGGESTED LEARNING STRATEGIES:** Group Presentation, Self Revision/Peer Revision, Look for a Pattern, Quickwrite

Perform addition of complex numbers as you would for addition of binomials of the form  $a + bx$ . To add such binomials, you collect like terms.

**Example A**

		Addition of Binomials	Addition of Complex Numbers
		$(5 + 4x) + (-2 + 3x)$	$(5 + 4i) + (-2 + 3i)$
<b>Step 1</b>	Collect like terms.	$= (5 - 2) + (4x + 3x)$	$= (5 - 2) + (4i + 3i)$
<b>Step 2</b>	Simplify.	$= 3 + 7x$	$= 3 + 7i$

**Try These A**

Add the complex numbers.

- $(6 + 5i) + (4 - 7i)$
- $(-5 + 3i) + (-3 - i)$
- $(2 + 3i) + (-2 - 3i)$

**1. Express regularity in repeated reasoning.** Use Example A above and your knowledge of operations of real numbers to write general formulas for the sum and difference of two complex numbers.

$$(a + bi) + (c + di) =$$

$$(a + bi) - (c + di) =$$





## My Notes

## MATH TIP

Since  $i = \sqrt{-1}$ , the powers of  $i$  can be evaluated as follows:

$$i^1 = i$$

$$i^2 = -1$$

$$i^3 = i^2 \cdot i = -1i = -i$$

$$i^4 = i^2 \cdot i^2 = (-1)^2 = 1$$

Since  $i^4 = 1$ , further powers repeat the pattern shown above.

$$i^5 = i^4 \cdot i = i$$

$$i^6 = i^4 \cdot i^2 = i^2 = -1$$

$$i^7 = i^4 \cdot i^3 = i^3 = -i$$

$$i^8 = i^4 \cdot i^4 = i^4 = 1$$

6. **Express regularity in repeated reasoning.** Use Example B and your knowledge of operations of real numbers to write a general formula for the multiplication of two complex numbers.

$$(a + bi) \cdot (c + di) =$$

7. Use operations of complex numbers to verify that the two solutions that Cardano found,  $x = 5 + \sqrt{-15}$  and  $x = 5 - \sqrt{-15}$ , have a sum of 10 and a product of 40.

## Check Your Understanding

8. Find each product.
- a.  $(5 + 3i)(5 - 3i)$       b.  $(-6 - 4i)(-6 + 4i)$   
 c.  $(8 + i)(8 - i)$
9. What patterns do you observe in the products in Item 8?
10. Explain how the product of two complex numbers can be a real number, even though both factors are not real numbers.
11. **Critique the reasoning of others.** A student claims that the product of any two imaginary numbers is a real number. Is the student correct? Explain your reasoning.

## Lesson 8-2

### Operations with Complex Numbers

## ACTIVITY 8

continued

The **complex conjugate** of  $a + bi$  is defined as  $a - bi$ . For example, the complex conjugate of  $2 + 3i$  is  $2 - 3i$ .

12. A special property of multiplication of complex numbers occurs when a number is multiplied by its conjugate. Multiply each number by its conjugate and then describe the product when a number is multiplied by its conjugate.

a.  $2 - 9i$

b.  $-5 + 2i$

13. Write an expression to complete the general formula for the product of a complex number and its complex conjugate.

$$(a + bi)(a - bi) =$$

To divide two complex numbers, start by multiplying both the dividend and the divisor by the conjugate of the divisor. This step results in a divisor that is a real number.

### Example C

Divide  $\frac{4 - 5i}{2 + 3i}$ .

**Step 1:** Multiply the numerator and denominator by the complex conjugate of the divisor.

$$\frac{4 - 5i}{2 + 3i} = \frac{4 - 5i}{2 + 3i} \cdot \frac{(2 - 3i)}{(2 - 3i)}$$

**Step 2:** Simplify and substitute  $-1$  for  $i^2$ .

$$\begin{aligned} &= \frac{8 - 22i + 15i^2}{4 - 6i + 6i - 9i^2} \\ &= \frac{8 - 22i - 15}{4 + 9} \end{aligned}$$

**Step 3:** Simplify and write in the form  $a + bi$ .

$$= \frac{-7 - 22i}{13} = -\frac{7}{13} - \frac{22}{13}i$$

**Solution:**  $\frac{4 - 5i}{2 + 3i} = -\frac{7}{13} - \frac{22}{13}i$

My Notes

### MATH TERMS

The **complex conjugate** of a complex number  $a + bi$  is  $a - bi$ .

My Notes

TECHNOLOGY TIP

Many graphing calculators have the capability to perform operations on complex numbers.

Try These C

a. In Example C, why is the quotient  $-\frac{7}{13} - \frac{22}{13}i$  equivalent to the original expression  $\frac{4-5i}{2+3i}$ ?

Divide the complex numbers. Write your answers on notebook paper. Show your work.

- b.  $\frac{5i}{2+3i}$       c.  $\frac{5+2i}{3-4i}$       d.  $\frac{1-i}{\sqrt{3}+4i}$

14. **Express regularity in repeated reasoning.** Use Example C and your knowledge of operations of real numbers to write a general formula for the division of two complex numbers.

$$\frac{(a+bi)}{(c+di)} =$$

Check Your Understanding

15. Make a conjecture about the quotient of two imaginary numbers where the divisor is not equal to  $0i$ . Is the quotient real, imaginary, or neither? Give an example to support your conjecture.
16. Make a conjecture about the quotient of a real number divided by an imaginary number not equal to  $0i$ . Is the quotient real, imaginary, or neither? Give an example to support your conjecture.
17. Which of the following is equal to  $i^{-1}$ ?  
 A. 1      B.  $-1$       C.  $i$       D.  $-i$
18. Explain your reasoning for choosing your answer to Item 17.

MATH TIP

For Item 17,  $n^{-1} = \frac{1}{n}$  for  $n \neq 0$ .  
 So,  $i^{-1} = \frac{1}{i}$ .



## My Notes

**Learning Targets:**

- Factor quadratic expressions using complex conjugates.
- Solve quadratic equations with complex roots by factoring.

**SUGGESTED LEARNING STRATEGIES:** Discussion Groups, Look for a Pattern, Quickwrite, Self Revision/Peer Revision, Paraphrasing

1. Look back at your answer to Item 13 in the previous lesson.
  - a. Given your answer, what are the factors of the expression  $a^2 + b^2$ ? Justify your answer.
  
  - b. What is the relationship between the factors of  $a^2 + b^2$ ?

You can use complex conjugates to factor quadratic expressions that can be written in the form  $a^2 + b^2$ . In other words, you can use complex conjugates to factor the sum of two squares.

2. **Express regularity in repeated reasoning.** Use complex conjugates to factor each expression.
  - a.  $16x^2 + 25$
  
  - b.  $36x^2 + 100y^2$
  
  - c.  $2x^2 + 8y^2$
  
  - d.  $3x^2 + 20y^2$

**MATH TIP**

You can check your answers to Item 2 by multiplying the factors. Check that the product is equal to the original expression.







**ACTIVITY 8 PRACTICE**

Write your answers on notebook paper. Show your work.

**Lesson 8-1**

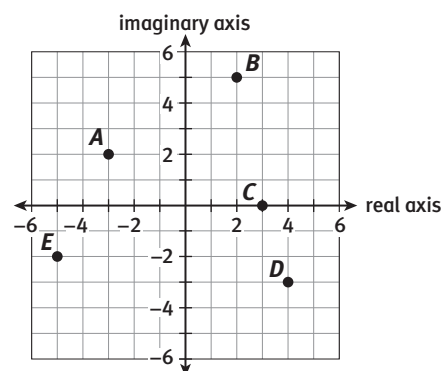
- Write each expression in terms of  $i$ .
  - $\sqrt{-64}$
  - $\sqrt{-31}$
  - $-7 + \sqrt{-12}$
  - $5 - \sqrt{-50}$
- Which expression is equivalent to  $5i$ ?
 

A. $\sqrt{-5}$	B. $-\sqrt{5}$
C. $\sqrt{-25}$	D. $-\sqrt{25}$
- Use the Quadratic Formula to solve each equation.
  - $x^2 + 5x + 9 = 0$
  - $2x^2 - 4x + 5 = 0$
- The sum of two numbers is 12, and their product is 100.
  - Let  $x$  represent one of the numbers. Write an expression for the other number in terms of  $x$ . Use the expressions to write an equation that models the situation given above.
  - Use the Quadratic Formula to solve the equation. Write the solutions in terms of  $i$ .
- Explain why each of the following is a complex number, and identify its real part and its imaginary part.
 

a. $5 + 3i$	b. $\sqrt{2} - i$
c. $-14i$	d. $\frac{3}{4}$
- Draw the complex plane on grid paper. Then graph each complex number on the plane.
 

a. $-4i$	b. $6 + 2i$
c. $-3 - 4i$	d. $3 - 5i$
e. $-2 + 5i$	

- What complex number does the ordered pair  $(5, -3)$  represent on the complex plane? Explain.
- Name the complex number represented by each labeled point on the complex plane below.



**Lesson 8-2**

- Find each sum or difference.
  - $(5 - 6i) + (-3 + 9i)$
  - $(2 + 5i) + (-5 + 3i)$
  - $(9 - 2i) - (1 + 6i)$
  - $(-5 + 4i) - \left(\frac{7}{3} + \frac{1}{6}i\right)$
- Find each product, and write it in the form  $a + bi$ .
  - $(1 + 4i)(5 - 2i)$
  - $(-2 + 3i)(3 - 2i)$
  - $(7 + 24i)(7 - 24i)$
  - $(8 - 3i)(4 - 2i)$
- Find each quotient, and write it in the form  $a + bi$ .
 

a. $\frac{3 + 2i}{5 - 2i}$	b. $\frac{-1 + i}{5 - 2i}$
c. $\frac{10 - 2i}{5i}$	d. $\frac{3 + i}{3 - i}$
- Explain how to use the Commutative, Associative, and Distributive Properties to perform each operation.
  - Subtract  $(3 + 4i)$  from  $(8 + 5i)$ .
  - Multiply  $(-2 + 3i)$  and  $(4 - 6i)$ .

**ACTIVITY 8**

continued

**Introduction to Complex Numbers**  
**Cardano's Imaginary Numbers**

13. Give an example of a complex number you could add to  $4 - 8i$  that would result in an imaginary number. Show that the sum of the complex numbers is equal to an imaginary number.
14. What is the complex conjugate of  $-3 + 7i$ ?  
**A.**  $-3 - 7i$                       **B.**  $3 - 7i$   
**C.**  $3 + 7i$                          **D.**  $7 - 3i$
15. Simplify each expression.  
**a.**  $-i^2$                               **b.**  $-6i^4$   
**c.**  $(2i)^3$                           **d.**  $\left(\frac{3}{2i^3}\right)^2$
16. What is the difference of any complex number  $a + bi$  and its complex conjugate?
17. Use substitution to show that the solutions of the equation  $x^2 - 6x + 34 = 0$  are  $x = 3 + 5i$  and  $x = 3 - 5i$ .
18. **a.** Graph the complex number  $4 + 2i$  on a complex plane.  
**b.** Multiply  $4 + 2i$  by  $i$ , and graph the result.  
**c.** Multiply the result from part b by  $i$ , and graph the result.  
**d.** Multiply the result from part c by  $i$ , and graph the result.  
**e.** Describe any patterns you see in the complex numbers you graphed.  
**f.** What happens when you multiply a complex number  $a + bi$  by  $i$ ?

**Lesson 8-3**

19. Use complex conjugates to factor each expression.  
**a.**  $x^2 + 121$                          **b.**  $2x^2 + 128y^2$   
**c.**  $4x^2 + 60y^2$                     **d.**  $9x^2 + 140y^2$
20. Explain how to solve the equation  $2x^2 + 100 = 0$  by factoring.
21. Solve each equation by factoring.  
**a.**  $x^2 + 64 = 0$                       **b.**  $x^2 = -120$   
**c.**  $4x^2 + 169 = 0$                  **d.**  $25x^2 = -48$
22. Which equation has solutions of  $x = -\frac{2}{3}i$  and  $x = \frac{2}{3}i$ ?  
**A.**  $3x^2 - 2 = 0$                       **B.**  $3x^2 + 2 = 0$   
**C.**  $9x^2 - 4 = 0$                       **D.**  $9x^2 + 4 = 0$
23. What are the solutions of each quadratic function?  
**a.**  $f(x) = x^2 + 1$   
**b.**  $f(x) = 25x^2 + 36$
24. Without solving the equation, explain how you know that  $x^2 + 48 = 0$  has imaginary solutions.

**MATHEMATICAL PRACTICES****Look for and Express Regularity in Repeated Reasoning**

25. Find the square of each complex number.  
**a.**  $(4 + 5i)$   
**b.**  $(2 + 3i)$   
**c.**  $(4 - 2i)$   
**d.** Use parts a–c and your knowledge of operations of real numbers to write a general formula for the square of a complex number  $(a + bi)$ .

# Solving $ax^2 + bx + c = 0$

## Deriving the Quadratic Formula

### Lesson 9-1 Completing the Square and Taking Square Roots

#### Learning Targets:

- Solve quadratic equations by taking square roots.
- Solve quadratic equations  $ax^2 + bx + c = 0$  by completing the square.

**SUGGESTED LEARNING STRATEGIES:** Marking the Text, Group Presentation, Quickwrite, Create Representations

To solve equations of the form  $ax^2 + c = 0$ , isolate  $x^2$  and take the square root of both sides of the equation.

#### Example A

Solve  $5x^2 - 23 = 0$  for  $x$ .

**Step 1:** Add 23 to both sides.

$$5x^2 - 23 = 0$$
$$5x^2 = 23$$

**Step 2:** Divide both sides by 5.

$$\frac{5x^2}{5} = \frac{23}{5}$$

**Step 3:** Simplify to isolate  $x^2$ .

$$x^2 = \frac{23}{5}$$

**Step 4:** Take the square root of both sides.

$$x = \pm \frac{\sqrt{23}}{\sqrt{5}}$$

**Step 5:** Rationalize the denominator.

$$x = \pm \frac{\sqrt{23}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$$

**Step 6:** Simplify.

$$x = \pm \frac{\sqrt{115}}{5}$$

**Solution:**  $x = \pm \frac{\sqrt{115}}{5}$

#### Try These A

**Make use of structure.** Solve for  $x$ . Show your work.

a.  $9x^2 - 49 = 0$

b.  $25x^2 - 7 = 0$

c.  $5x^2 - 16 = 0$

d.  $4x^2 + 15 = 0$

My Notes

#### MATH TIP

When taking the square root of both sides of an equation, include both positive and negative roots. For example,

$$x^2 = 4$$
$$x = \pm\sqrt{4}$$
$$x = \pm 2$$

#### MATH TIP

To rewrite an expression so that there are no radicals in the denominator, you must *rationalize the denominator* by multiplying both the numerator and denominator by the radical.

Example:

$$\frac{7}{\sqrt{3}} = \frac{7}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{7\sqrt{3}}{3}$$

## My Notes

## CONNECT TO AP

In calculus, *rationalizing a numerator* is a skill used to evaluate certain types of limit expressions.

1. Compare and contrast the solutions to the equations in Try These A.

To solve the equation  $2(x - 3)^2 - 5 = 0$ , you can use a similar process.

**Example B**

Solve  $2(x - 3)^2 - 5 = 0$  for  $x$ .

**Step 1:** Add 5 to both sides.

$$2(x - 3)^2 - 5 = 0$$

$$2(x - 3)^2 = 5$$

**Step 2:** Divide both sides by 2.

$$(x - 3)^2 = \frac{5}{2}$$

**Step 3:** Take the square root of both sides.

$$x - 3 = \pm \frac{\sqrt{5}}{\sqrt{2}}$$

**Step 4:** Rationalize the denominator and solve for  $x$ .

$$x - 3 = \pm \frac{\sqrt{10}}{2}$$

**Solution:**  $x = 3 \pm \frac{\sqrt{10}}{2}$

**Try These B**

Solve for  $x$ . Show your work.

a.  $4(x + 5)^2 - 49 = 0$

b.  $3(x - 2)^2 - 16 = 0$

c.  $5(x + 1)^2 - 8 = 0$

d.  $4(x + 7)^2 + 25 = 0$

2. **Reason quantitatively.** Describe the differences among the solutions to the equations in Try These B.

## Lesson 9-1

### Completing the Square and Taking Square Roots

## ACTIVITY 9

continued

### Check Your Understanding

- Use Example A to help you write a general formula for the solutions of the equation  $ax^2 - c = 0$ , where  $a$  and  $c$  are both positive.
- Is the equation solved in Example B a quadratic equation? Explain.
- Solve the equation  $-2(x + 4)^2 + 3 = 0$ , and explain each of your steps.
- Solve the equation  $3(x - 5)^2 = 0$ .
  - Make use of structure.** Explain why the equation has only one solution and not two solutions.

The standard form of a quadratic equation is  $ax^2 + bx + c = 0$ . You can solve equations written in standard form by **completing the square**.

### Example C

Solve  $2x^2 + 12x + 5 = 0$  by completing the square.

**Step 1:** Divide both sides by the leading coefficient and simplify.

$$2x^2 + 12x + 5 = 0$$

$$\frac{2x^2}{2} + \frac{12x}{2} + \frac{5}{2} = \frac{0}{2}$$

$$x^2 + 6x + \frac{5}{2} = 0$$

**Step 2:** Isolate the variable terms on the left side.

$$x^2 + 6x = -\frac{5}{2}$$

**Step 3:** Divide the coefficient of the linear term by 2 [ $6 \div 2 = 3$ ], square the result [ $3^2 = 9$ ], and add it [9] to both sides. This completes the square.

$$x^2 + 6x + \square = -\frac{5}{2} + \square$$

$$x^2 + 6x + \boxed{9} = -\frac{5}{2} + \boxed{9}$$

**Step 4:** Factor the perfect square trinomial on the left side into two binomials.

$$(x + 3)^2 = \frac{13}{2}$$

**Step 5:** Take the square root of both sides of the equation.

$$x + 3 = \pm\sqrt{\frac{13}{2}}$$

**Step 6:** Rationalize the denominator and solve for  $x$ .

$$x + 3 = \pm\frac{\sqrt{13}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \pm\frac{\sqrt{26}}{2}$$

**Solution:**  $x = -3 \pm \frac{\sqrt{26}}{2}$

### My Notes

### MATH TERMS

**Completing the square** is the process of adding a constant to a quadratic expression to transform it into a perfect square trinomial.

### MATH TIP

You can factor a perfect square trinomial  $x^2 + 2xy + y^2$  as  $(x + y)^2$ .



## My Notes

## Try These C

Solve for  $x$  by completing the square.

a.  $4x^2 + 16x - 5 = 0$

b.  $5x^2 - 30x - 3 = 0$

c.  $2x^2 - 6x - 1 = 0$

d.  $2x^2 - 4x + 7 = 0$

## Check Your Understanding

7. Explain how to complete the square for the quadratic expression  $x^2 + 8x$ .
8. How does completing the square help you solve a quadratic equation?
9. **Construct viable arguments.** Which method would you use to solve the quadratic equation  $x^2 + x - 12 = 0$ : factoring or completing the square? Justify your choice.

## LESSON 9-1 PRACTICE

10. Use the method for completing the square to make a perfect square trinomial. Then factor the perfect square trinomial.
  - a.  $x^2 + 10x$
  - b.  $x^2 - 7x$
11. Solve each quadratic equation by taking the square root of both sides of the equation. Identify the solutions as rational, irrational, or complex conjugates.
  - a.  $9x^2 - 64 = 0$
  - b.  $5x^2 - 12 = 0$
  - c.  $16(x - 2)^2 - 25 = 0$
  - d.  $2(x - 3)^2 - 15 = 0$
  - e.  $4x^2 + 49 = 0$
  - f.  $3(x - 1)^2 + 10 = 0$
12. Solve by completing the square.
  - a.  $x^2 - 4x - 12 = 0$
  - b.  $2x^2 - 5x - 3 = 0$
  - c.  $x^2 + 6x - 2 = 0$
  - d.  $3x^2 + 9x + 2 = 0$
  - e.  $x^2 - x + 5 = 0$
  - f.  $5x^2 + 2x + 3 = 0$
13. The diagonal of a rectangular television screen measures 42 in. The ratio of the length to the width of the screen is  $\frac{16}{9}$ .
  - a. **Model with mathematics.** Write an equation that can be used to determine the length  $l$  in inches of the television screen.
  - b. Solve the equation, and interpret the solutions.
  - c. What are the length and width of the television screen, to the nearest half-inch?

## CONNECT TO GEOMETRY

The length, width, and diagonal of the television screen form a right triangle.

**Learning Targets:**

- Derive the Quadratic Formula.
- Solve quadratic equations using the Quadratic Formula.

**SUGGESTED LEARNING STRATEGIES:** Create Representations, Discussion Groups, Self Revision/Peer Revision, Think-Pair-Share, Quickwrite

Previously you learned that solutions to the general quadratic equation  $ax^2 + bx + c = 0$  can be found using the Quadratic Formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ where } a \neq 0$$

You can *derive* the quadratic formula by completing the square on the general quadratic equation.

- 1. Reason abstractly and quantitatively.** Derive the quadratic formula by completing the square for the equation  $ax^2 + bx + c = 0$ . (Use Example C from Lesson 9-1 as a model.)

My Notes

**ACADEMIC VOCABULARY**

When you *derive* a formula, you use logical reasoning to show that the formula is correct. In this case, you will derive the Quadratic Formula by solving the standard form of a quadratic equation,  $ax^2 + bx + c = 0$ , for  $x$ .

## My Notes

## ACADEMIC VOCABULARY

When you **verify** a solution, you check that it is correct.

2. Solve  $2x^2 - 5x + 3 = 0$  by completing the square. Then **verify** that the solution is correct by solving the same equation using the Quadratic Formula.

## Check Your Understanding

3. In Item 1, why do you need to add  $\left(\frac{b}{2a}\right)^2$  to both sides?
4. Derive a formula for solving a quadratic equation of the form  $ax^2 + bx = 0$ , where  $a \neq 0$ .
5. **Construct viable arguments.** Which method did you prefer for solving the quadratic equation in Item 2: completing the square or using the Quadratic Formula? Justify your choice.
6. Consider the equation  $x^2 - 6x + 7 = 0$ .
- Solve the equation by using the Quadratic Formula.
  - Could you have solved the equation by factoring? Explain.

LESSON 9-2 PRACTICE

7. Solve each equation by using the Quadratic Formula.
- a.  $2x^2 + 4x - 5 = 0$
  - b.  $3x^2 + 7x + 10 = 0$
  - c.  $x^2 - 9x - 1 = 0$
  - d.  $-4x^2 + 5x + 8 = 0$
  - e.  $2x^2 - 3 = 7x$
  - f.  $4x^2 + 3x = -6$
8. Solve each quadratic equation by using any of the methods you have learned. For each equation, tell which method you used and why you chose that method.
- a.  $x^2 + 6x + 9 = 0$
  - b.  $8x^2 + 5x - 6 = 0$
  - c.  $(x + 4)^2 - 36 = 0$
  - d.  $x^2 + 2x = 7$
9. a. **Reason abstractly.** Under what circumstances will the radicand in the Quadratic Formula,  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ , be negative?
- b. If the radicand is negative, what does this tell you about the solutions of the quadratic equation? Explain.
10. A player shoots a basketball from a height of 7 ft with an initial vertical velocity of 18 ft/s. The equation  $-16t^2 + 18t + 7 = 10$  can be used to determine the time  $t$  in seconds at which the ball will have a height of 10 ft—the same height as the basket.
- a. Solve the equation by using the Quadratic Formula.
  - b. **Attend to precision.** To the nearest tenth of a second, when will the ball have a height of 10 ft?
  - c. Explain how you can check that your answers to part b are reasonable.

My Notes

MATH TIP

A *radicand* is an expression under a radical symbol. For  $\sqrt{b^2 - 4ac}$ , the radicand is  $b^2 - 4ac$ .

CONNECT TO PHYSICS

The function  $h(t) = -16t^2 + v_0t + h_0$  can be used to model the height  $h$  in feet of a thrown object  $t$  seconds after it is thrown, where  $v_0$  is the initial vertical velocity of the object in ft/s and  $h_0$  is the initial height of the object in feet.

## My Notes

## MATH TIP

The complex numbers include the real numbers, so real solutions are also complex solutions. However, when asked to classify solutions as real or complex, you can assume that “complex” does not include the reals.

**Learning Targets:**

- Solve quadratic equations using the Quadratic Formula.
- Use the discriminant to determine the nature of the solutions of a quadratic equation.

**SUGGESTED LEARNING STRATEGIES:** Look for a Pattern, Group Presentation, Self Revision/Peer Revision, Think-Pair-Share, Quickwrite

1. Solve each equation by using the Quadratic Formula. For each equation, write the number of solutions. Tell whether the solutions are real or complex, and, if real, whether the solutions are rational or irrational.

a.  $4x^2 + 5x - 6 = 0$

solutions:

number of solutions:

real or complex:

rational or irrational:

b.  $4x^2 + 5x - 2 = 0$

solutions:

number of solutions:

real or complex:

rational or irrational:

c.  $4x^2 + 4x + 1 = 0$

solutions:

number of solutions:

real or complex:

rational or irrational:

d.  $4x^2 + 4x + 5 = 0$

solutions:

number of solutions:

real or complex:

rational or irrational:



## My Notes

## MATH TERMS

The **discriminant** is the expression  $b^2 - 4ac$  under the radical sign in the Quadratic Formula.

## MATH TERMS

A solution to an equation is also called a **root** of the equation.

The roots of a quadratic equation  $ax^2 + bx + c = 0$  represent the **zeros** (or  $x$ -intercepts) of the quadratic function  $y = ax^2 + bx + c$ .

## MATH TIP

If the values of  $a$ ,  $b$ , and  $c$  are integers and the discriminant  $b^2 - 4ac$  is a perfect square, then the quadratic expression  $ax^2 + bx + c$  is factorable over the integers.

The **discriminant** of a quadratic equation  $ax^2 + bx + c = 0$  is defined as the expression  $b^2 - 4ac$ . The value of the discriminant determines the *nature of the solutions* of a quadratic equation in the following manner.

Discriminant	Nature of Solutions
$b^2 - 4ac > 0$ and $b^2 - 4ac$ is a perfect square	Two real, rational solutions
$b^2 - 4ac > 0$ and $b^2 - 4ac$ is <i>not</i> a perfect square	Two real, irrational solutions
$b^2 - 4ac = 0$	One real, rational solution (a double <b>root</b> )
$b^2 - 4ac < 0$	Two complex conjugate solutions

7. Compute the value of the discriminant for each equation in Item 1 to determine the number and nature of the solutions.

a.  $4x^2 + 5x - 6 = 0$

b.  $4x^2 + 5x - 2 = 0$

c.  $4x^2 + 4x + 1 = 0$

d.  $4x^2 + 4x + 5 = 0$





## My Notes

## LESSON 9-3 PRACTICE

13. For each equation, evaluate the discriminant and determine the nature of the solutions. Then solve each equation using the Quadratic Formula to verify the nature of the roots.

a.  $x^2 + 5x - 6 = 0$

b.  $2x^2 - 7x - 15 = 0$

c.  $x^2 - 8x + 16 = 0$

d.  $5x^2 - 4x + 2 = 0$

e.  $2x^2 + 9x + 20 = 0$

f.  $3x^2 - 5x - 1 = 0$

14. **Reason abstractly.** What is the discriminant? How does the value of the discriminant affect the solutions of a quadratic equation?

15. The discriminant of a quadratic equation is 1. What can you conclude about the solutions of the equation? Explain your reasoning.

16. Give an example of a quadratic equation that has two irrational solutions. Use the discriminant to show that the solutions of the equation are irrational.

17. **Make sense of problems.** A baseball player throws a ball from a height of 6 ft with an initial vertical velocity of 32 ft/s. The equation  $-16t^2 + 32t + 6 = 25$  can be used to determine the time  $t$  in seconds at which the ball will reach a height of 25 ft.

- a. Evaluate the discriminant of the equation.

- b. What does the discriminant tell you about whether the ball will reach a height of 25 ft?

## MATH TIP

In Item 17, remember to write the equation in standard form before you evaluate the discriminant.

**ACTIVITY 9 PRACTICE**

Write your answers on notebook paper.  
 Show your work.

**Lesson 9-1**

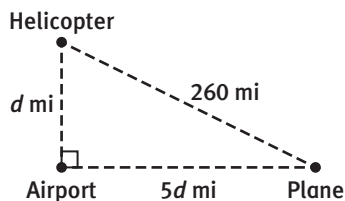
For Items 1–8, solve each equation by taking the square root of both sides.

1.  $4x^2 - 49 = 0$                       2.  $5x^2 = 36$   
 3.  $9x^2 - 32 = 0$                       4.  $(x + 4)^2 - 25 = 0$   
 5.  $3(x + 2)^2 = 15$                     6.  $-2(x - 4)^2 = 16$   
 7.  $4(x - 8)^2 - 10 = 14$             8.  $6(x + 3)^2 + 20 = 12$

9. Which of the following represents a formula that can be used to solve quadratic equations of the form  $a(x - h)^2 + k = 0$ , where  $a \neq 0$ ?

- A.  $x = -h \pm \sqrt{-\frac{k}{a}}$                       B.  $x = -h \pm \sqrt{\frac{k}{a}}$   
 C.  $x = h \pm \sqrt{-\frac{k}{a}}$                       D.  $x = h \pm \sqrt{\frac{k}{a}}$

10. A plane begins flying due east from an airport at the same time as a helicopter begins flying due north from the airport. After half an hour, the plane and helicopter are 260 mi apart, and the plane is five times the distance from the airport as the helicopter.



Not to scale

- a. Write an equation that can be used to determine  $d$ , the helicopter's distance in miles from the airport after half an hour.  
 b. Solve the equation and interpret the solutions.  
 c. What are the average speeds of the plane and the helicopter? Explain.

For Items 11–14, complete the square for each quadratic expression. Then factor the perfect square trinomial.

11.  $x^2 + 10x$                               12.  $x^2 - 16x$   
 13.  $x^2 + 9x$                               14.  $x^2 - x$

For Items 15–20, solve each equation by completing the square.

15.  $x^2 + 2x + 5 = 0$   
 16.  $x^2 - 10x = 26$   
 17.  $x^2 + 5x - 9 = 0$   
 18.  $2x^2 + 8x - 7 = 0$   
 19.  $3x^2 - 15x = 20$   
 20.  $6x^2 + 16x + 9 = 0$

**Lesson 9-2**

For Items 21–28, solve each equation by using the Quadratic Formula.

21.  $x^2 + 12x + 6 = 0$   
 22.  $3x^2 - 5x + 3 = 0$   
 23.  $2x^2 + 6x = 25$   
 24.  $42x^2 + 11x - 20 = 0$   
 25.  $x^2 + 6x + 8 = 4x - 3$   
 26.  $10x^2 - 5x = 9x + 8$   
 27.  $4x^2 + x - 12 = 3x^2 - 5x$   
 28.  $x^2 - 20x = 6x^2 - 2x + 20$

29. Write a formula that represents the solutions of a quadratic equation of the form  $mx^2 + nx + p = 0$ . Explain how you arrived at your formula.  
 30. Derive a formula for solving a quadratic equation of the form  $x^2 + bx + c = 0$ .

For Items 31–36, solve each equation, using any method that you choose. For each equation, tell which method you used and why you chose that method.

31.  $(x + 3)^2 - 25 = 0$

32.  $2x^2 - 9x + 5 = 0$

33.  $x^2 + 7x + 12 = 0$

34.  $3x^2 + x - 14 = 0$

35.  $x^2 + 8x = 7$

36.  $4x^2 - 33 = 0$

37. The more concert tickets a customer buys, the less each individual ticket costs. The function  $c(t) = -2t^2 + 82t + 5$  gives the total cost in dollars of buying  $t$  tickets to the concert.

Customers may buy no more than 15 tickets.

- Megan spent a total of \$301 on concert tickets. Write a quadratic equation that can be used to determine the number of tickets Megan bought.
- Use the Quadratic Formula to solve the equation. Then interpret the solutions.
- What was the cost of each ticket Megan bought?

### Lesson 9-3

For each equation, find the value of the discriminant and describe the nature of the solutions.

38.  $2x^2 + 3x + 4 = 0$

39.  $9x^2 + 30x + 25 = 0$

40.  $6x^2 - 7x - 20 = 0$

41.  $5x^2 + 12x - 7 = 0$

42.  $x^2 - 8x = 18$

43. The discriminant of a quadratic equation is  $-6$ . What types of solutions does the equation have?

- 1 real solution
- 2 rational solutions
- 2 irrational solutions
- 2 complex conjugate solutions

44. Consider the quadratic equation  $ax^2 - 6x + 3 = 0$ , where  $a \neq 0$ .

- For what value(s) of  $a$  does the equation have two real solutions?
- For what value(s) of  $a$  does the equation have one real solution?
- For what value(s) of  $a$  does the equation have two complex conjugate solutions?

45. The function  $p(s) = -14s^2 + 440s - 2100$  models the monthly profit in dollars made by a small T-shirt company when the selling price of its shirts is  $s$  dollars.

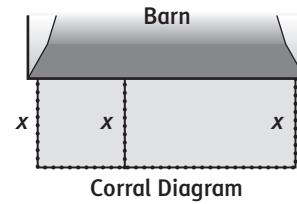
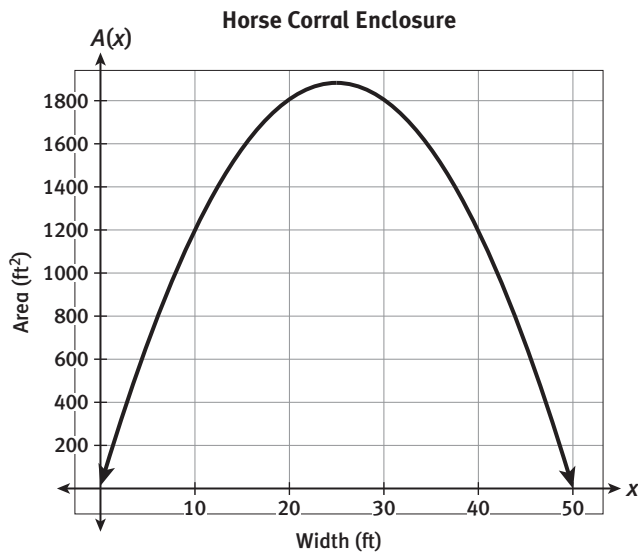
- Write an equation that can be used to determine the selling price that will result in a monthly profit of \$1200.
- Evaluate the discriminant of the equation.
- What does the discriminant tell you about whether the company can have a monthly profit of \$1200?

### MATHEMATICAL PRACTICES

#### Look for and Make Use of Structure

46. Tell which method you would use to solve each quadratic equation having the given form. Then explain why you would use that method.

- $ax^2 + c = 0$
- $ax^2 + bx = 0$
- $x^2 + bx = -c$ , where  $b$  is even
- $x^2 + bx + c = 0$ , where  $c$  has a factor pair with a sum of  $b$
- $ax^2 + bx + c = 0$ , where  $a$ ,  $b$ , and  $c$  are each greater than 10



- Kun-cha has 150 feet of fencing to make a corral for her horses. The barn will be one side of the partitioned rectangular enclosure, as shown in the diagram above. The graph illustrates the function that represents the area that could be enclosed.
  - Write a function,  $A(x)$ , that represents the area that can be enclosed by the corral.
  - What information does the graph provide about the function?
  - Which ordered pair indicates the maximum area possible for the corral? Explain what each coordinate tells about the problem.
  - What values of  $x$  will give a total area of  $1000 \text{ ft}^2$ ?  $2000 \text{ ft}^2$ ?
- Critique the reasoning of others.** Tim is the punter for the Bitterroot Springs Mustangs football team. He wrote a function  $h(t) = 16t^2 + 8t + 1$  that he thinks will give the height of a football in terms of  $t$ , the number of seconds after he kicks the ball. Use two different methods to determine the values of  $t$  for which  $h(t) = 0$ . Show your work. Is Tim's function correct? Why or why not?
- Tim has been studying complex numbers and quadratic equations. His teacher, Mrs. Pinto, gave the class a quiz. Demonstrate your understanding of the material by responding to each item below.
  - Write a quadratic equation that has two solutions,  $x = 2 + 5i$  and  $x = 2 - 5i$ .
  - Solve  $3x^2 + 2x - 8 = 0$ , using an algebraic method.
  - Rewrite  $\frac{4+i}{3-2i}$  in the form  $a + bi$ , where  $a$  and  $b$  are rational numbers.

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
<p><b>Mathematics Knowledge and Thinking</b> (Items 1c, 1d, 2, 3a-c)</p>	<ul style="list-style-type: none"> <li>Effective understanding of and accuracy in solving quadratic equations algebraically or graphically</li> <li>Clear and accurate understanding of the key features of graphs of quadratic functions and the relationship between zeros and solutions to quadratic equations</li> <li>Clear and accurate understanding of how to perform operations with complex numbers</li> </ul>	<ul style="list-style-type: none"> <li>Adequate understanding of solving quadratic equations algebraically or graphically, leading to solutions that are usually correct</li> <li>Largely correct understanding of the key features of graphs of quadratic functions and the relationship between zeros and solutions to quadratic equations</li> <li>Largely correct understanding of how to perform operations with complex numbers</li> </ul>	<ul style="list-style-type: none"> <li>Partial understanding of and some difficulty solving quadratic equations algebraically or graphically</li> <li>Partial understanding of the key features of graphs of quadratic functions and the relationship between zeros and solutions to quadratic equations</li> <li>Difficulty performing operations with complex numbers</li> </ul>	<ul style="list-style-type: none"> <li>Inaccurate or incomplete understanding of solving quadratic equations algebraically or graphically</li> <li>Little or no understanding of the key features of graphs of quadratic functions and the relationship between zeros and solutions to quadratic equations</li> <li>Little or no understanding of how to perform operations with complex numbers</li> </ul>
<p><b>Problem Solving</b> (Items 1c, 1d, 2)</p>	<ul style="list-style-type: none"> <li>An appropriate and efficient strategy that results in a correct answer</li> </ul>	<ul style="list-style-type: none"> <li>A strategy that may include unnecessary steps but results in a correct answer</li> </ul>	<ul style="list-style-type: none"> <li>A strategy that results in some incorrect answers</li> </ul>	<ul style="list-style-type: none"> <li>No clear strategy when solving problems</li> </ul>
<p><b>Mathematical Modeling / Representations</b> (Item 1)</p>	<ul style="list-style-type: none"> <li>Effective understanding of how to write a quadratic equation or function from a verbal description, graph or diagram</li> <li>Clear and accurate understanding of how to interpret features of the graphs of quadratic functions and the solutions to quadratic equations</li> </ul>	<ul style="list-style-type: none"> <li>Adequate understanding of how to write a quadratic equation or function from a verbal description, graph or diagram</li> <li>Largely correct understanding of how to interpret features of the graphs of quadratic functions and the solutions to quadratic equations</li> </ul>	<ul style="list-style-type: none"> <li>Partial understanding of how to write a quadratic equation or function from a verbal description, graph or diagram</li> <li>Some difficulty with interpreting the features of graphs of quadratic functions and the solutions to quadratic equations</li> </ul>	<ul style="list-style-type: none"> <li>Little or no understanding of how to write a quadratic equation or function from a verbal description, graph or diagram</li> <li>Inaccurate or incomplete interpretation of the features of graphs of quadratic functions and the solutions to quadratic equations</li> </ul>
<p><b>Reasoning and Communication</b> (Items 1b, 1c, 2)</p>	<ul style="list-style-type: none"> <li>Precise use of appropriate math terms and language to relate equations and graphs of quadratic functions and their key features to a real-world scenario</li> <li>Clear and accurate use of mathematical work to justify or refute a claim</li> </ul>	<ul style="list-style-type: none"> <li>Adequate descriptions to relate equations and graphs of quadratic functions and their key features to a real-world scenario</li> <li>Correct use of mathematical work to justify or refute a claim</li> </ul>	<ul style="list-style-type: none"> <li>Misleading or confusing descriptions to relate equations and graphs of quadratic functions and their key features to a real-world scenario</li> <li>Partially correct use of mathematical work to justify or refute a claim</li> </ul>	<ul style="list-style-type: none"> <li>Incomplete or inaccurate descriptions to relate equations and graphs of quadratic functions and their key features to a real-world scenario</li> <li>Incorrect or incomplete use of mathematical work to justify or refute a claim</li> </ul>







## My Notes

## TECHNOLOGY TIP

If an equation includes the  $\pm$  symbol, you will need to enter it in a graphing calculator as two separate equations. For example, enter the equation  $y = 2 \pm \sqrt{x}$  as  $y = 2 + \sqrt{x}$  and  $y = 2 - \sqrt{x}$ .

2. Explain how you matched each equation with one of the graphs.

3. **Use appropriate tools strategically.** Use a graphing calculator to confirm your answers to Item 1. Which equations must be rewritten to enter them in the calculator? Rewrite any equations from Item 1 as necessary so that you can use them with your calculator.

4. a. How do graphs A and B differ from graphs C and D?

b. How do the equations of graphs A and B differ from the equations of graphs C and D?

## Lesson 10-1

### Parabolas and Quadratic Equations

## ACTIVITY 10

continued

5. Work with your group. Consider graphs A and B and their equations.
- Describe the relationship between the graphs.
  - What part of the equation determines whether the graph opens up or down? How do you know?
  - Attend to precision.** What are the coordinates of the lowest point on graph A? What are the coordinates of the highest point on graph B? How do the coordinates of these points relate to the equations of the graphs?
6. Continue to work with your group. Consider graphs C and D and their equations.
- Describe the relationship between the graphs.
  - What part of the equation determines whether the graph opens to the right or left? How do you know?
  - What are the coordinates of the leftmost point on graph C? What are the coordinates of the rightmost point on graph D? How do the coordinates of these points relate to the equations of the graphs?

My Notes

### DISCUSSION GROUP TIP

As you share ideas for Items 5 and 6 in your group, ask your group members or your teacher for clarification of any language, terms, or concepts that you do not understand.

### MATH TIP

A graph is said to open upward when both ends of the graph point up. A graph is said to open downward when both ends of the graph point down.

The vertex of a graph that opens upward is the *minimum* of the graph, and is its lowest point. The vertex of a graph that opens downward is the *maximum* of the graph, and is its highest point.

My Notes

Check Your Understanding

7. Which equation does the graph at right represent? Explain your answer.

A.  $y = -\frac{1}{2}(x + 2)^2 - 4$

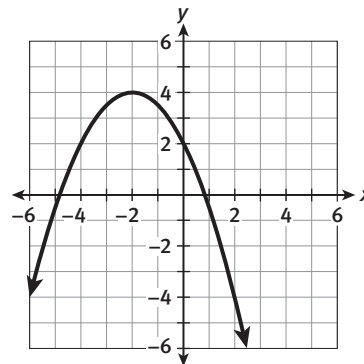
B.  $y = -\frac{1}{2}(x + 2)^2 + 4$

C.  $y = -\frac{1}{2}(x - 2)^2 + 4$

8. **Construct viable arguments.**

Which of the equations in Item 1 represent functions? Explain your reasoning.

9. Consider the equation  $x = -2(y + 4)^2 - 1$ . Without graphing the equation, tell which direction its graph opens. Explain your reasoning.



MATH TERMS

A **parabola** is the set of points in a plane that are equidistant from a fixed point and a fixed line.

The fixed point is called the **focus**.

The fixed line is called the **directrix**.

MATH TIP

The distance between two points  $(x_1, y_1)$  and  $(x_2, y_2)$  is given by

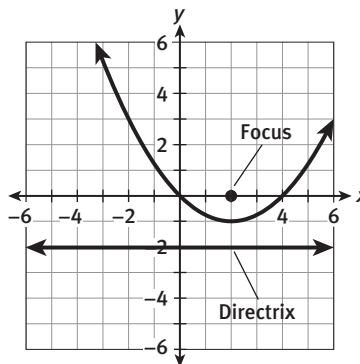
$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

MATH TIP

The distance between a point and a horizontal line is the length of the vertical segment with one endpoint at the point and one endpoint on the line.

The graphs shown at the beginning of this lesson are all parabolas. A **parabola** can be defined as the set of points that are the same distance from a point called the **focus** and a line called the **directrix**.

10. The focus of graph A, shown below, is  $(2, 0)$ , and the directrix is the horizontal line  $y = -2$ .



a. The point  $(-2, 3)$  is on the parabola. Find the distance between this point and the focus.

## Lesson 10-1

### Parabolas and Quadratic Equations

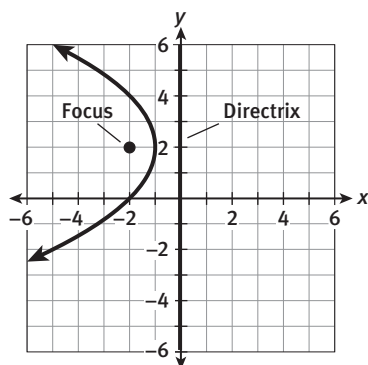
## ACTIVITY 10

continued

b. Find the distance between the point  $(-2, 3)$  and the directrix.

c. **Reason quantitatively.** Compare your answers in parts a and b. What do you notice?

11. The focus of graph D, shown below, is  $(-2, 2)$ , and the directrix is the vertical line  $x = 0$ .



a. The point  $(-2, 4)$  is on the parabola. Show that this point is the same distance from the focus as from the directrix.

b. The point  $(-5, -2)$  is also on the parabola. Show that this point is the same distance from the focus as from the directrix.

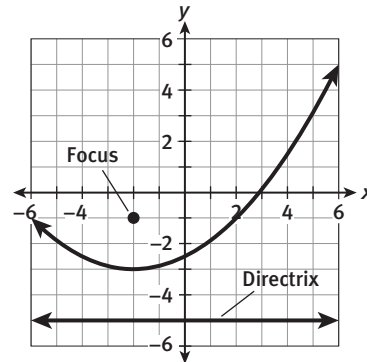
My Notes

### MATH TIP

The distance between a point and a vertical line is the length of the horizontal segment with one endpoint at the point and one endpoint on the line.

My Notes

The focus of the parabola shown below is  $(-2, -1)$ , and the directrix is the line  $y = -5$ .



MATH TERMS

The **axis of symmetry** is a line that divides the parabola into two congruent halves. The axis of symmetry passes through the focus and is perpendicular to the directrix.

The **vertex** is the point on the parabola that lies on the axis of symmetry. The vertex is the midpoint of the segment connecting the focus and the directrix.

12. a. Draw and label the **axis of symmetry** on the graph above. What is the equation of the axis of symmetry?

b. Explain how you identified the axis of symmetry of the parabola.

13. a. Draw and label the **vertex** on the graph above. What are the coordinates of the vertex?

b. Explain how you identified the vertex of the parabola.

c. What is another way you could have identified the vertex?

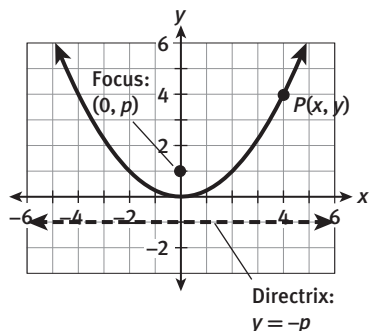
## Lesson 10-1

### Parabolas and Quadratic Equations

## ACTIVITY 10

continued

You can use what you have learned about parabolas to derive a general equation for a parabola whose vertex is located at the origin. Start with a parabola that has a vertical axis of symmetry, a focus of  $(0, p)$ , and a directrix of  $y = -p$ . Let  $P(x, y)$  represent any point on the parabola.



- Write, but do not simplify, an expression for the distance from point  $P$  to the focus.
- Write, but do not simplify, an expression for the distance from point  $P$  to the directrix.
- Make use of structure.** Based on the definition of a parabola, the distance from point  $P$  to the focus is the same as the distance from point  $P$  to the directrix. Set your expressions from Items 14 and 15 equal to each other, and then solve for  $y$ .

My Notes

### MATH TIP

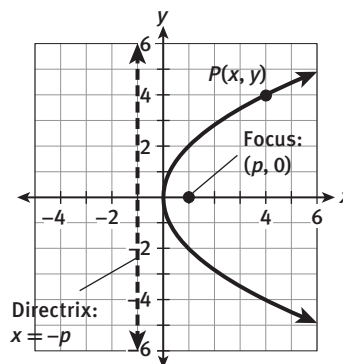
In Item 16, start by squaring each side of the equation to eliminate the square root symbols. Next, simplify each side and expand the squared terms.

**My Notes**

17. What is the general equation for a parabola with its vertex at the origin, a focus of  $(0, p)$ , and a directrix of  $y = -p$ ?

**Check Your Understanding**

18. See the diagram at right. Derive the general equation of a parabola with its vertex at the origin, a horizontal axis of symmetry, a focus of  $(p, 0)$ , and a directrix of  $x = -p$ . Solve the equation for  $x$ .



19. **Model with mathematics.** The vertex of a parabola is at the origin and its focus is  $(0, -3)$ . What is the equation of the parabola? Explain your reasoning.
20. A parabola has a focus of  $(3, 4)$  and a directrix of  $x = -1$ . Answer each question about the parabola, and explain your reasoning.
- What is the axis of symmetry?
  - What is the vertex?
  - In which direction does the parabola open?

**MATH TIP**

A parabola always opens toward the focus and away from the directrix.

You can also write general equations for parabolas that do not have their vertex at the origin. You will derive these equations later in this activity.

	Vertical Axis of Symmetry	Horizontal Axis of Symmetry
<b>Vertex</b>	$(h, k)$	$(h, k)$
<b>Focus</b>	$(h, k + p)$	$(h + p, k)$
<b>Directrix</b>	horizontal line $y = k - p$	vertical line $x = h - p$
<b>Equation</b>	$y = \frac{1}{4p}(x - h)^2 + k$	$x = \frac{1}{4p}(y - k)^2 + h$



## Lesson 10-1

### Parabolas and Quadratic Equations

## ACTIVITY 10

*continued*

**21. Reason quantitatively.** Use the given information to write the equation of each parabola.

**a.** axis of symmetry:  $y = 0$ ; vertex:  $(0, 0)$ ; directrix:  $x = \frac{1}{2}$

**b.** vertex:  $(3, 4)$ ; focus:  $(3, 6)$

**c.** vertex:  $(-2, 1)$ ; directrix:  $y = 4$

**d.** focus:  $(-4, 0)$ ; directrix:  $x = 4$

**e.** opens up; focus:  $(5, 7)$ ; directrix:  $y = 3$

My Notes

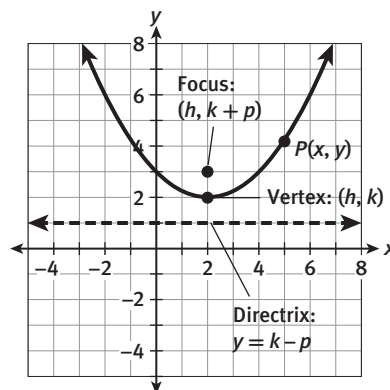
### MATH TIP

You may find it helpful to make a quick sketch of the information you are given.

My Notes

Check Your Understanding

22. See the diagram at right. Derive the general equation of a parabola with its vertex at  $(h, k)$ , a vertical axis of symmetry, a focus of  $(h, k + p)$ , and a directrix of  $y = k - p$ . Solve the equation for  $y$ .

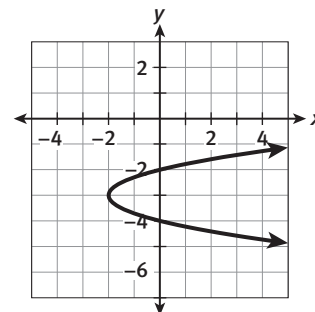


23. **Construct viable arguments.** Can you determine the equation of a parabola if you know only its axis of symmetry and its vertex? Explain.

24. The equation of a parabola is  $x = \frac{1}{8}(y - 2)^2 + 1$ . Identify the vertex, axis of symmetry, focus, and directrix of the parabola.

LESSON 10-1 PRACTICE

25. Which equation does the graph at right represent?  
 A.  $x = -2(y + 3)^2 - 2$   
 B.  $x = 2(y + 3)^2 - 2$   
 C.  $y = -2(x + 3)^2 - 2$   
 D.  $y = 2(x + 3)^2 - 2$



26. Graph the parabola given by the equation  $y = \frac{1}{4}(x + 3)^2 - 4$ .

27. **Make sense of problems.** The focus of a parabola is  $(0, 2)$ , and its directrix is the vertical line  $x = -6$ . Identify the axis of symmetry, the vertex, and the direction the parabola opens.

Use the given information to write the equation of each parabola.

28. vertex:  $(0, 0)$ ; focus:  $(0, -\frac{1}{2})$   
 29. focus:  $(4, 0)$ ; directrix:  $x = -4$   
 30. opens to the left; vertex:  $(0, 5)$ ; focus:  $(-5, 5)$   
 31. axis of symmetry:  $x = 3$ ; focus:  $(3, -1)$ ; directrix:  $y = -7$   
 32. vertex:  $(-2, 4)$ ; directrix:  $x = -3$

**Learning Targets:**

- Explain why three points are needed to determine a parabola.
- Determine the quadratic function that passes through three given points on a plane.

**SUGGESTED LEARNING STRATEGIES:** Create Representations, Quickwrite, Questioning the Text, Create Representations, Identify a Subtask

Recall that if you are given any two points on the coordinate plane, you can write the equation of the line that passes through those points. The two points are said to determine the line because there is only one line that can be drawn through them.

Do two points on the coordinate plane determine a parabola? To answer this question, work through the following items.

1. Follow these steps to write the equation of a quadratic function whose graph passes through the points  $(2, 0)$  and  $(5, 0)$ .
  - a. Write a quadratic equation in standard form with the solutions  $x = 2$  and  $x = 5$ .
  - b. Replace 0 in your equation from part a with  $y$  to write the corresponding quadratic function.
  - c. Use substitution to check that the points  $(2, 0)$  and  $(5, 0)$  lie on the function's graph.
2. **a. Use appropriate tools strategically.** Graph your quadratic function from Item 1 on a graphing calculator.
  - b. On the same screen, graph the quadratic functions  $y = 2x^2 - 14x + 20$  and  $y = -x^2 + 7x - 10$ .

My Notes

**MATH TIP**

To review writing a quadratic equation when given its solutions, see Lesson 7-3.

## My Notes

- c. Describe the graphs. Do all three parabolas pass through the points (2, 0) and (5, 0)?

3. **Reason abstractly.** Do two points on the coordinate plane determine a parabola? Explain.

## MATH TIP

Three or more points are *collinear* if they lie on the same straight line.

Three points in the coordinate plane that are not on the same line determine a parabola given by a quadratic function. If you are given three noncollinear points on the coordinate plane, you can write the equation of the quadratic function whose graph passes through them.

Consider the quadratic function whose graph passes through the points (1, 2), (3, 0), and (5, 6).

4. Write an equation by substituting the coordinates of the point (1, 2) into the standard form of a quadratic function,  $y = ax^2 + bx + c$ .
5. Write a second equation by substituting the coordinates of the point (3, 0) into the standard form of a quadratic function.
6. Write a third equation by substituting the coordinates of the point (5, 6) into the standard form of a quadratic function.

## Lesson 10-2

### Writing a Quadratic Function Given Three Points

## ACTIVITY 10

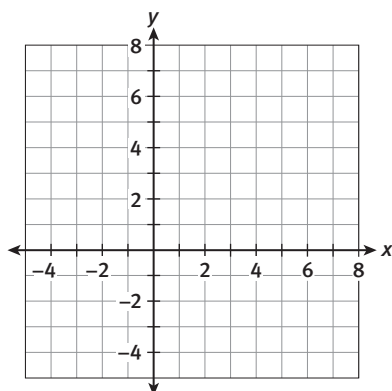
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7. Use your equations from Items 4–6 to write a system of three equations in the three variables  $a$ ,  $b$ , and  $c$ .

8. Use substitution or Gaussian elimination to solve your system of equations for  $a$ ,  $b$ , and  $c$ .

9. Now substitute the values of  $a$ ,  $b$ , and  $c$  into the standard form of a quadratic function.

10. **Model with mathematics.** Graph the quadratic function to confirm that it passes through the points  $(1, 2)$ ,  $(3, 0)$ , and  $(5, 6)$ .



My Notes

### MATH TIP

To review solving a system of three equations in three variables, see Lesson 3-2.

My Notes

Check Your Understanding

11. Describe how to write the equation of a quadratic function whose graph passes through three given points.
12. a. What happens when you try to write the equation of the quadratic function that passes through the points  $(0, 4)$ ,  $(2, 2)$ , and  $(4, 0)$ ?  
b. What does this result indicate about the three points?
13. a. **Reason quantitatively.** The graph of a quadratic function passes through the point  $(2, 0)$ . The vertex of the graph is  $(-2, -16)$ . Use symmetry to identify another point on the function's graph. Explain how you determined your answer.  
b. Write the equation of the quadratic function.

LESSON 10-2 PRACTICE

Write the equation of the quadratic function whose graph passes through each set of points.

14.  $(-3, 2)$ ,  $(-1, 0)$ ,  $(1, 6)$
15.  $(-2, -5)$ ,  $(0, -3)$ ,  $(1, 4)$
16.  $(-1, -5)$ ,  $(1, -9)$ ,  $(4, 0)$
17.  $(-3, 7)$ ,  $(0, 4)$ ,  $(1, 15)$
18.  $(1, 0)$ ,  $(2, -7)$ ,  $(5, -16)$
19.  $(-2, -11)$ ,  $(-1, -12)$ ,  $(1, 16)$
20. The table below shows the first few terms of a sequence. This sequence can be described by a quadratic function, where  $f(n)$  represents the  $n$ th term of the sequence. Write the quadratic function that describes the sequence.

Term Number, $n$	1	2	3	4	5
Term of Sequence, $f(n)$	2	6	12	20	30

21. A quadratic function  $A(s)$  gives the area in square units of a regular hexagon with a side length of  $s$  units.
  - a. Use the data in the table below to write the equation of the quadratic function.

Side Length, $s$	2	4	6
Area, $A(s)$	$6\sqrt{3}$	$24\sqrt{3}$	$54\sqrt{3}$

- b. **Attend to precision.** To the nearest square centimeter, what is the area of a regular hexagon with a side length of 8 cm?

MATH TIP

A sequence is an ordered list of numbers or other items. Each number or item in a sequence is called a term.

CONNECT TO GEOMETRY

A regular hexagon is a six-sided polygon with all sides having the same length and all angles having the same measure.





## My Notes

3. Enter the rocket data into a graphing calculator. Enter the time data as List 1 (L1) and the height data as List 2 (L2). Then use the calculator to perform a linear regression on the data. Write the equation of the linear model that results from the regression. Round coefficients and constants to the nearest tenth.
  
4. Use a dashed line to graph the linear model from Item 3 on the coordinate grid showing the rocket data.
  
5. **a. Attend to precision.** To the nearest meter, what height does the linear model predict for the rocket 12 seconds after it is launched?
  
- b.** How does this prediction compare with the prediction you made in Item 1?
  
6. **Construct viable arguments.** Do you think the linear model is a good model for the rocket data? Justify your answer.

## MATH TIP

A calculator may be able to generate a linear model for a data set, but that does not necessarily mean that the model is a good fit or makes sense in a particular situation.

## MATH TERMS

**Quadratic regression** is the process of determining the equation of a quadratic function that best fits the given data.

A linear regression is the process of finding a linear function that best fits a set of data. A **quadratic regression** is the process of finding a quadratic function that best fits a set of data. The steps for performing a quadratic regression on a graphing calculator are similar to those for performing a linear regression.

## Lesson 10-3

### Quadratic Regression

## ACTIVITY 10

continued

- Use these steps to perform a quadratic regression for the rocket data.
  - Check that the data set is still entered as List 1 and List 2.
  - Press  $\boxed{\text{STAT}}$  to select the Statistics menu. Then move the cursor to highlight the Calculate (CALC) submenu.
  - Select 5:QuadReg to perform a quadratic regression on the data in Lists 1 and 2. Press  $\boxed{\text{ENTER}}$ .
  - The calculator displays the values of  $a$ ,  $b$ , and  $c$  for the standard form of the quadratic function that best fits the data.

Write the equation of the quadratic model that results from the regression. Round coefficients and constants to the nearest tenth.

- Graph the quadratic model from Item 7 on the coordinate grid showing the rocket data.

- Construct viable arguments.** Contrast the graph of the linear model with the graph of the quadratic model. Which model is a better fit for the data?

- a.** To the nearest meter, what height does the quadratic model predict for the rocket 12 seconds after it is launched?

- b.** How does this prediction compare with the prediction you made in Item 1?

- Reason quantitatively.** Use the quadratic model to predict when the rocket will hit the ground. Explain how you determined your answer.

My Notes

### TECHNOLOGY TIP

You can graph the equation from a quadratic regression by using these steps: After selecting **5:QuadReg** as described at the left, do not press  $\boxed{\text{ENTER}}$ . Instead, press  $\boxed{\text{VARS}}$  to select the VARS menu. Then move the cursor to highlight the Y-VARS submenu. Select **1:Function**. Then select **1:Y1**. Press  $\boxed{\text{ENTER}}$ . The equation from the quadratic regression is now assigned to Y1. You can press  $\boxed{\text{GRAPH}}$  to view the graph of the equation.

My Notes

**Check Your Understanding**

- 12. Make sense of problems.** Most model rockets have a parachute or a similar device that releases shortly after the rocket reaches its maximum height. The parachute helps to slow the rocket so that it does not hit the ground with as much force. Based on this information, do you think your prediction from Item 11 is an underestimate or an overestimate if the rocket has a parachute? Explain.
- 13. a.** Could you use a graphing calculator to perform a quadratic regression on three data points? Explain.  
**b.** How closely would the quadratic model fit the data set in this situation? Explain.  
**c.** How would your answers to parts a and b change if you knew that the three points lie on the same line?

**LESSON 10-3 PRACTICE**

Tell whether a linear model or a quadratic model is a better fit for each data set. Justify your answer, and give the equation of the better model.

**14.**

<b>x</b>	10	12	14	16	18	20	22	24
<b>y</b>	19	15	13	11	9	9	10	11

**15.**

<b>x</b>	2	4	6	8	10	12	14	16
<b>y</b>	10	22	26	35	45	50	64	66

The tables show time and height data for two other model rockets.

<b>Rocket A</b>	<b>Time (s)</b>	0	1	2	3	4	5	6	7
	<b>Height (m)</b>	0	54	179	255	288	337	354	368

<b>Rocket B</b>	<b>Time (s)</b>	0	1	2	3	4	5	6	7
	<b>Height (m)</b>	0	37	92	136	186	210	221	229

- 16. Use appropriate tools strategically.** Use a graphing calculator to perform a quadratic regression for each data set. Write the equations of the quadratic models. Round coefficients and constants to the nearest tenth.
- 17.** Use your models to predict which rocket had a greater maximum height. Explain how you made your prediction.
- 18.** Use your models to predict which rocket hit the ground first and how much sooner. Explain how you made your prediction.

# Writing Quadratic Equations

## What Goes Up Must Come Down

### ACTIVITY 10

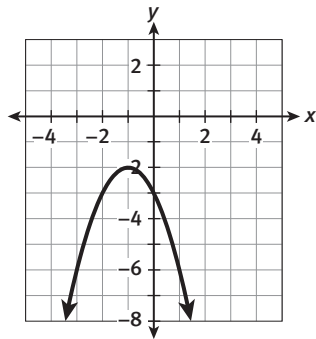
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## ACTIVITY 10 PRACTICE

Write your answers on notebook paper.  
Show your work.

### Lesson 10-1

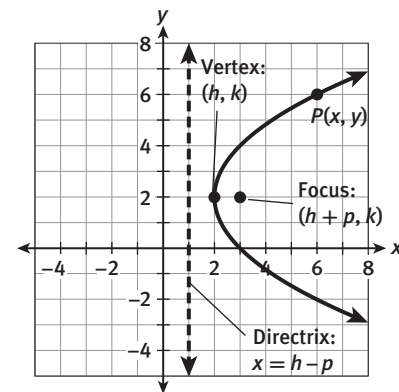
Use the parabola shown in the graph for Items 1 and 2.



- What is the equation of the parabola?
  - $y = -(x - 1)^2 - 2$
  - $y = -(x + 1)^2 - 2$
  - $y = (x - 1)^2 - 2$
  - $y = (x + 1)^2 + 2$
- The focus of the parabola is  $(-1, -\frac{9}{4})$ , and the directrix is the line  $y = -\frac{7}{4}$ . Show that the point  $(-2, -3)$  on the parabola is the same distance from the focus as from the directrix.
- Graph the parabola given by the equation  $x = \frac{1}{2}(y - 3)^2 + 3$ .
- Identify the following features of the parabola given by the equation  $y = \frac{1}{8}(x - 4)^2 + 3$ .
  - vertex
  - focus
  - directrix
  - axis of symmetry
  - direction of opening
- Describe the relationships among the vertex, focus, directrix, and axis of symmetry of a parabola.
- The focus of a parabola is  $(3, -2)$ , and its directrix is the line  $x = -5$ . What are the vertex and the axis of symmetry of the parabola?

For Items 7–11, use the given information to write the equation of each parabola.

- vertex:  $(0, 0)$ ; focus:  $(0, 5)$
- vertex:  $(0, 0)$ ; directrix:  $x = -3$
- vertex:  $(2, 2)$ ; axis of symmetry:  $y = 2$ ; focus:  $(1, 2)$
- opens downward; vertex:  $(-1, -2)$ ; directrix:  $y = -1$
- focus:  $(-1, 3)$ ; directrix:  $x = -5$
- Use the diagram below to help you derive the general equation of a parabola with its vertex at  $(h, k)$ , a horizontal axis of symmetry, a focus of  $(h + p, k)$ , and a directrix of  $x = h - p$ . Solve the equation for  $x$ .



### Lesson 10-2

Write the equation of the quadratic function whose graph passes through each set of points.

- $(-3, 0), (-2, -3), (2, 5)$
- $(-2, -6), (1, 0), (2, 10)$
- $(-5, -3), (-4, 0), (0, -8)$
- $(-3, 10), (-2, 0), (0, -2)$
- $(1, 0), (4, 6), (7, -6)$
- $(-2, -9), (-1, 0), (1, -12)$

**ACTIVITY 10***continued***Writing Quadratic Equations****What Goes Up Must Come Down**

19. Demonstrate that the points  $(-8, 0)$  and  $(6, 0)$  do not determine a unique parabola by writing the equations of two different parabolas that pass through these two points.
20. a. The graph of a quadratic function passes through the point  $(7, 5)$ . The vertex of the graph is  $(3, 1)$ . Use symmetry to identify another point on the function's graph. Explain your answer.  
b. Write the equation of the quadratic function.
24. Use your models to predict how much farther it would take the truck to stop from a speed of 50 mi/h than it would the car.
25. Suppose the truck is 300 ft from an intersection when the light at the intersection turns yellow. If the truck's speed is 60 mi/h when the driver sees the light change, will the driver be able to stop without entering the intersection? Explain how you know.

**Lesson 10-3**

Tell whether a linear model or a quadratic model is a better fit for each data set. Justify your answer and give the equation of the better model.

21. 

<i>x</i>	0	2	4	6	8	10	12	14
<i>y</i>	17	29	40	45	59	63	76	88

22. 

<i>x</i>	2	4	6	8	10	12	14	16
<i>y</i>	15	9	5	2	6	7	16	22

The stopping distance of a vehicle is the distance the vehicle travels between the time the driver recognizes the need to stop and the time the vehicle comes to a stop. The table below shows how the speed of two vehicles affects their stopping distances.

Speed (mi/h)	Stopping distance (ft)	
	Car	Truck
10	27	28
15	44	47
20	63	69
25	85	95
30	109	123
35	135	155
40	164	190

23. Use a graphing calculator to perform a quadratic regression on the data for each vehicle. Write the equations of the quadratic models. Round coefficients and constants to the nearest thousandth.

**MATHEMATICAL PRACTICES****Use Appropriate Tools Strategically**

26. A shoe company tests different prices of a new type of athletic shoe at different stores. The table shows the relationship between the selling price and the monthly revenue per store the company made from selling the shoes.

Selling Price (\$)	Monthly Revenue per Store (\$)
80	9680
90	10,520
100	11,010
110	10,660
120	10,400
130	9380

- a. Use a graphing calculator to determine the equation of a quadratic model that can be used to predict  $y$ , the monthly revenue per store in dollars when the selling price is  $x$  dollars. Round values to the nearest tenth.
- b. Is a quadratic model a good model for the data set? Explain.
- c. Use your model to determine the price at which the company should sell the shoes to generate the greatest revenue.

# Transformations of $y = x^2$

## Parent Parabola

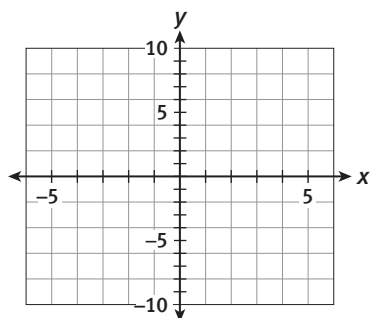
### Lesson 11-1 Translations of Parabolas

#### Learning Targets:

- Describe translations of the parent function  $f(x) = x^2$ .
- Given a translation of the function  $f(x) = x^2$ , write the equation of the function.

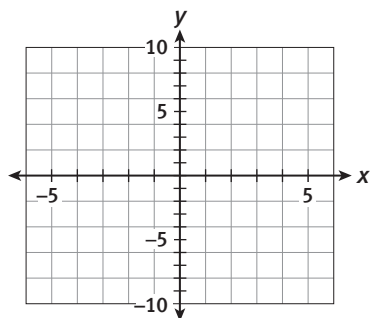
**SUGGESTED LEARNING STRATEGIES:** Create Representations, Quickwrite, Group Presentation, Look for a Pattern, Discussion Groups

1. Graph the parent quadratic function,  $f(x) = x^2$ , on the coordinate grid below. Include the points that have  $x$ -values  $-2, -1, 0, 1,$  and  $2$ .



The points on the parent function graph that have  $x$ -values  $-2, -1, 0, 1,$  and  $2$  are *key points* that can be used when graphing any quadratic function as a transformation of the parent quadratic function.

2. Graph  $f(x) = x^2$  on the coordinate grid below. Then graph and label  $g(x) = x^2 - 3$  and  $h(x) = x^2 + 2$ .



3. **Make use of structure.** Identify and describe the transformations of the graph of  $f(x) = x^2$  that result in the graphs of  $g(x)$  and  $h(x)$ .

#### My Notes

#### MATH TIP

A *parent function* is the simplest function of a particular type. For example, the parent linear function is  $f(x) = x$ . The parent absolute value function is  $f(x) = |x|$ .

#### MATH TIP

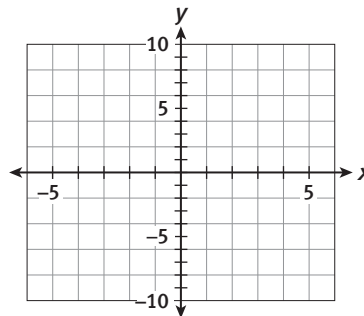
A *transformation* of a graph of a parent function is a change in the position, size, or shape of the graph.

**My Notes**

**MATH TIP**

*Translations* are transformations that change the location of a graph but maintain the original shape of a graph. For this reason, they are known as *rigid transformations*.

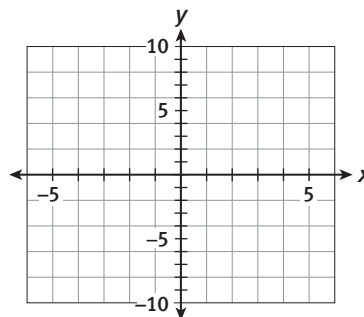
4. **Model with mathematics.** Graph  $f(x) = x^2$  on the coordinate grid below. Then graph and label  $g(x) = (x - 2)^2$  and  $h(x) = (x + 3)^2$ .



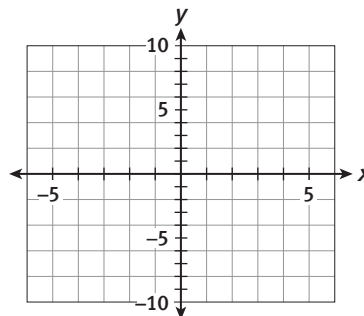
5. Identify and describe the transformations of the graph of  $f(x) = x^2$  that result in the graphs of  $g(x)$  and  $h(x)$ .

6. Describe each function as a transformation of  $f(x) = x^2$ . Then use that information to graph each function on the coordinate grid.

a.  $a(x) = (x - 1)^2$



b.  $w(x) = x^2 + 4$





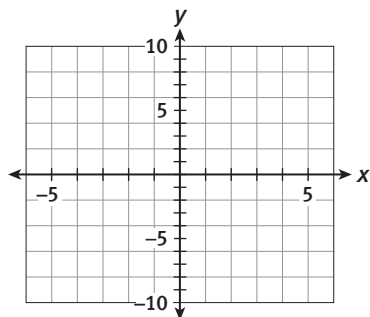
## Lesson 11-1

### Translations of Parabolas

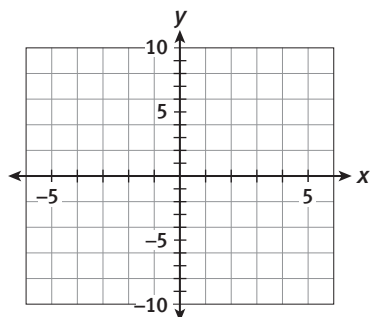
## ACTIVITY 11

continued

c.  $d(x) = (x + 3)^2 - 5$



d.  $j(x) = (x - 1)^2 + 2$



My Notes

### Check Your Understanding

**7. Express regularity in repeated reasoning.** The graph of each function below is a translation of the graph of  $f(x) = x^2$  by  $k$  units, where  $k > 0$ . For each function, tell which direction the graph of  $f(x)$  is translated.

a.  $g(x) = x^2 + k$

b.  $h(x) = (x + k)^2$

c.  $j(x) = x^2 - k$

d.  $m(x) = (x - k)^2$

**8.** What is the vertex of the function  $p(x) = x^2 - 5$ ? Justify your answer in terms of a translation of  $f(x) = x^2$ .

**9.** What is the axis of symmetry of the function  $q(x) = (x + 1)^2$ ? Justify your answer in terms of a translation of  $f(x) = x^2$ .

**10. Reason abstractly.** The function  $r(x)$  is a translation of the function  $f(x) = x^2$ . What can you conclude about the direction in which the parabola given by  $r(x)$  opens? Justify your answer.

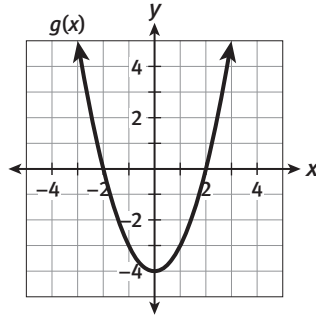
### MATH TIP

If you need help with Item 7, try substituting a positive number for  $k$  and then graphing each function.

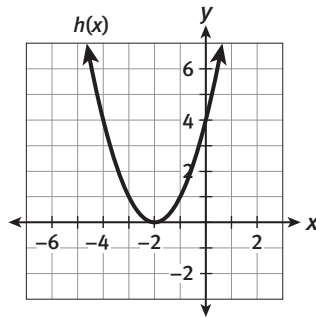
**My Notes**

- 11.** Each function graphed below is a translation of  $f(x) = x^2$ . Describe the transformation. Then write the equation of the transformed function.

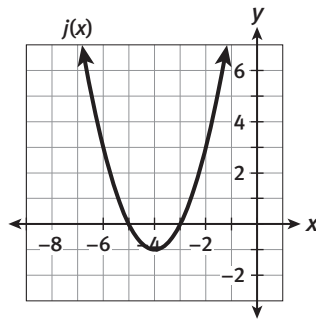
**a.**



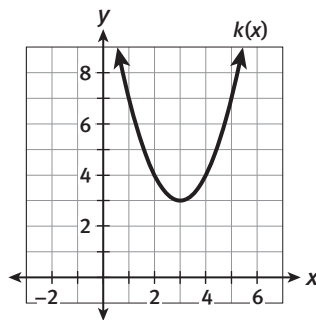
**b.**



**c.**



**d.**



## Lesson 11-1

### Translations of Parabolas

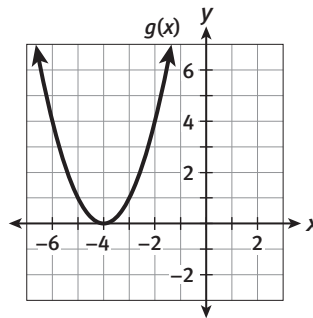
## ACTIVITY 11

continued

12. Use a graphing calculator to graph each of the equations you wrote in Item 11. Check that the graphs on the calculator match those shown in Item 11. Revise your answers to Item 11 as needed.

### Check Your Understanding

13. Explain how you determined the equation of  $k(x)$  in Item 11d.
14. **Critique the reasoning of others.**  
The graph shows a translation of  $f(x) = x^2$ . A student says that the equation of the transformed function is  $g(x) = (x - 4)^2$ . Is the student correct? Explain.
15. The graph of  $h(x)$  is a translation of the graph of  $f(x) = x^2$ . If the vertex of the graph of  $h(x)$  is  $(-1, -2)$ , what is the equation of  $h(x)$ ? Explain your answer.



### My Notes

### TECHNOLOGY TIP

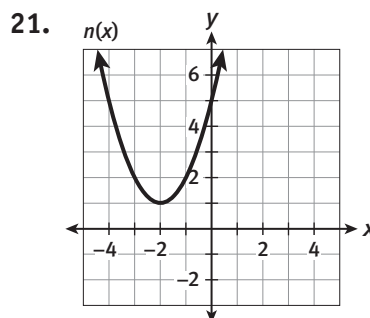
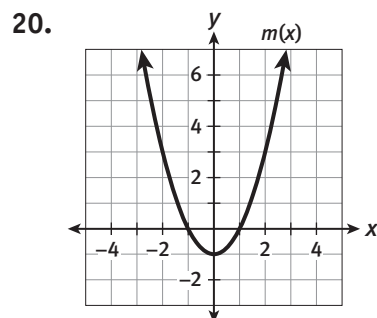
When you graph a function on a graphing calculator, the distance between tick marks on the  $x$ -axis is not always the same as the distance between tick marks on the  $y$ -axis. To make these distances the same, press **ZOOM**, and select **5 : ZSquare**. This step will make it easier to compare your calculator graphs to the graphs in Item 11.

## LESSON 11-1 PRACTICE

**Make sense of problems.** Describe each function as a transformation of  $f(x) = x^2$ .

16.  $g(x) = x^2 - 6$                       17.  $h(x) = (x + 5)^2$   
18.  $j(x) = (x - 2)^2 + 8$             19.  $k(x) = (x + 6)^2 - 4$

Each function graphed below is a translation of  $f(x) = x^2$ . Describe the transformation. Then write the equation of the transformed function.



22. What is the vertex of the function  $p(x) = (x - 5)^2 + 4$ ? Justify your answer in terms of a translation of  $f(x) = x^2$ .
23. What is the axis of symmetry of the function  $q(x) = (x + 8)^2 - 10$ ? Justify your answer in terms of a translation of  $f(x) = x^2$ .

## My Notes

## MATH TIP

Unlike a rigid transformation, a *vertical stretch* or *vertical shrink* will change the shape of the graph.

A vertical stretch stretches a graph away from the  $x$ -axis by a factor and a vertical shrink shrinks the graph toward the  $x$ -axis by a factor.

## MATH TIP

*Reflections* over axes do not change the shape of the graph, so they are also rigid transformations.

## Learning Targets:

- Describe transformations of the parent function  $f(x) = x^2$ .
- Given a transformation of the function  $f(x) = x^2$ , write the equation of the function.

**SUGGESTED LEARNING STRATEGIES:** Create Representations, Look for a Pattern, Group Presentation, Quickwrite, Identify a Subtask

1. Graph the function  $f(x) = x^2$  as Y1 on a graphing calculator. Then graph each of the following functions as Y2. Describe the graph of each function as a transformation of the graph of  $f(x) = x^2$ .

a.  $g(x) = 2x^2$

b.  $h(x) = 4x^2$

c.  $j(x) = \frac{1}{2}x^2$

d.  $k(x) = \frac{1}{4}x^2$

2. **Express regularity in repeated reasoning.** Describe any patterns you observed in the graphs from Item 1.

3. Graph the function  $f(x) = x^2$  as Y1 on a graphing calculator. Then graph each of the following functions as Y2. Identify and describe the graph of each function as a transformation of the graph of  $f(x) = x^2$ .
  - a.  $g(x) = -x^2$

## Lesson 11-2

### Shrinking, Stretching, and Reflecting Parabolas

## ACTIVITY 11

continued

b.  $h(x) = -4x^2$

c.  $j(x) = -\frac{1}{4}x^2$

4. Describe any patterns you observed in the graphs from Item 3.

5. Make a conjecture about how the sign of  $k$  affects the graph of  $g(x) = kx^2$  compared to the graph of  $f(x) = x^2$ . Assume that  $k \neq 0$ .

6. Make a conjecture about whether the absolute value of  $k$  affects the graph of  $g(x) = kx^2$  when compared to the graph of  $f(x) = x^2$ . Assume that  $k \neq 0$  and write your answer using absolute value notation.

7. **Make use of structure.** Without graphing, describe each function as a transformation of  $f(x) = x^2$ .

a.  $h(x) = 6x^2$

b.  $j(x) = -\frac{1}{10}x^2$

My Notes

### MATH TIP

In Item 6, consider the situation in which  $|k| > 1$  and the situation in which  $|k| < 1$ .

## My Notes

c.  $p(x) = -9x^2$

d.  $q(x) = \frac{1}{5}x^2$

## Check Your Understanding

8. The graph of  $g(x)$  is a vertical shrink of the graph of  $f(x) = x^2$  by a factor of  $\frac{1}{6}$ . What is the equation of  $g(x)$ ?
9. **Reason quantitatively.** The graph of  $h(x)$  is a vertical stretch of the graph of  $f(x) = x^2$ . If the graph of  $h(x)$  passes through the point  $(1, 7)$ , what is the equation of  $h(x)$ ? Explain your answer.
10. The graph of  $j(x) = kx^2$  opens downward. Based on this information, what can you conclude about the value of  $k$ ? Justify your conclusion.
11. Graph the function  $f(x) = x^2$  as Y1 on a graphing calculator. Then graph each of the following functions as Y2. Identify and describe the graph of each function as a horizontal stretch or shrink of the graph of  $f(x) = x^2$ .
- a.  $g(x) = (2x)^2$
- b.  $h(x) = (4x)^2$
- c.  $j(x) = \left(\frac{1}{2}x\right)^2$
- d.  $k(x) = \left(\frac{1}{4}x\right)^2$

## MATH TIP

A horizontal stretch stretches a graph away from the  $y$ -axis by a factor and a vertical shrink shrinks the graph toward the  $y$ -axis by a factor.

## Lesson 11-2

### Shrinking, Stretching, and Reflecting Parabolas

## ACTIVITY 11

continued

Work with your group on Items 12–16.

12. Describe any patterns you observed in the graphs from Item 11.
13. **a. Use appropriate tools strategically.** Graph the function  $f(x) = x^2$  as Y1 on a graphing calculator. Then graph  $h(x) = (-x)^2$  as Y2. Describe the result.
- b. Reason abstractly.** Explain why this result makes sense.
14. Make a conjecture about how the sign of  $k$  affects the graph of  $g(x) = (kx)^2$  compared to the graph of  $f(x) = x^2$ . Assume that  $k \neq 0$ .
15. Make a conjecture about whether the absolute value of  $k$  affects the graph of  $g(x) = (kx)^2$  when compared to the graph of  $f(x) = x^2$ . Assume that  $k \neq 0$ .
16. Describe each function as a transformation of  $f(x) = x^2$ .
- a.**  $p(x) = (6x)^2$
- b.**  $q(x) = \left(\frac{1}{10}x\right)^2$

My Notes

### DISCUSSION GROUP TIP

In your discussion groups, read the text carefully to clarify meaning. Reread definitions of terms as needed to help you comprehend the meanings of words, or ask your teacher to clarify vocabulary terms.

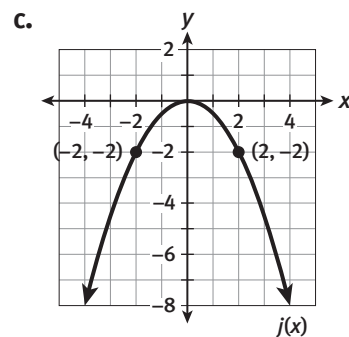
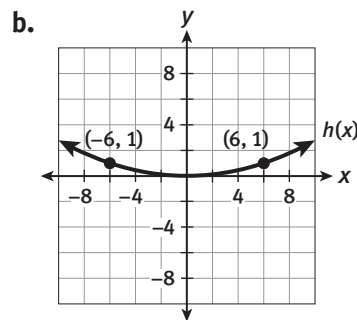
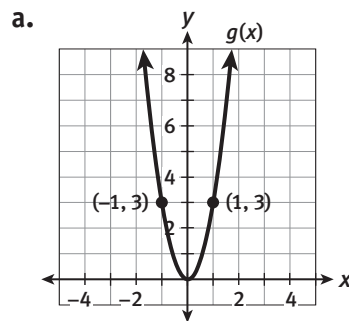


My Notes

Check Your Understanding

17. Describe how the graph of  $g(x) = 4x^2$  differs from the graph of  $h(x) = (4x)^2$ .
18. The graph of  $g(x)$  is a horizontal stretch of the graph of  $f(x) = x^2$  by a factor of 5. What is the equation of  $g(x)$ ?
19. **Reason quantitatively.** The graph of  $h(x)$  is a horizontal shrink of the graph of  $f(x) = x^2$ . If the graph of  $h(x)$  passes through the point  $(1, 25)$ , what is the equation of  $h(x)$ ? Explain your answer.

20. Each function graphed below is a transformation of  $f(x) = x^2$ . Describe the transformation. Then write the equation of the transformed function.

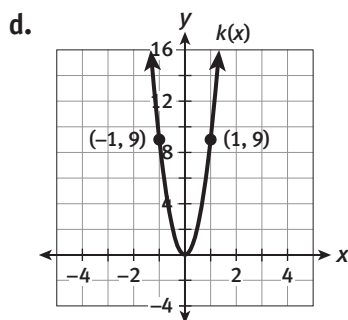


# Lesson 11-2

## Shrinking, Stretching, and Reflecting Parabolas

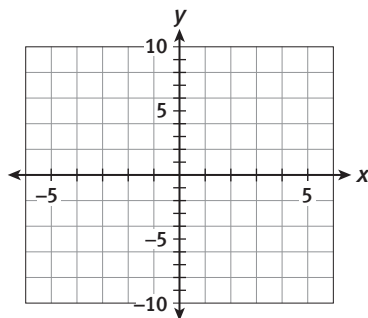
### ACTIVITY 11

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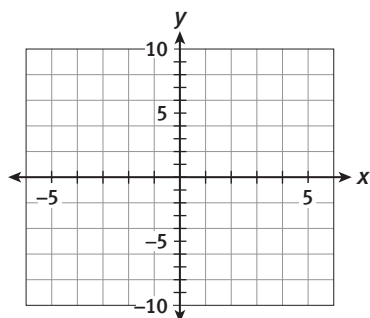


**21. Model with mathematics.** Multiple transformations can be represented in the same function. Describe the transformations from the parent function. Then graph the function, using your knowledge of transformations only.

a.  $f(x) = -4(x + 3)^2 + 2$



b.  $f(x) = 2(x - 4)^2 - 3$



My Notes

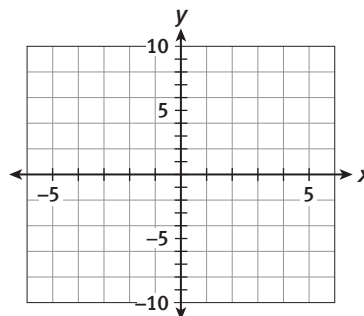
#### MATH TIP

When graphing multiple transformations of quadratic functions, follow this order:

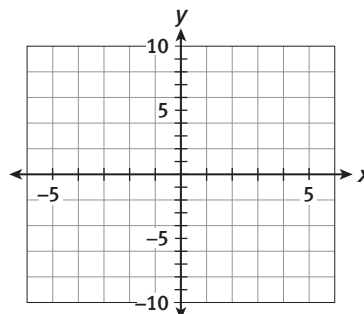
1. horizontal translation
2. horizontal shrink or stretch
3. reflection over the x-axis and/or vertical shrink or stretch
4. vertical translation

My Notes

c.  $f(x) = 2(x + 1)^2 - 4$



d.  $f(x) = -(x - 3)^2 + 5$



Check Your Understanding

22. Explain how you determined the equation of  $g(x)$  in Item 20a.
23. Without graphing, determine the vertex of the graph of  $h(x) = 2(x - 3)^2 + 4$ . Explain how you found your answer.
24.
  - a. Start with the graph of  $f(x) = x^2$ . Reflect it over the  $x$ -axis and then translate it 1 unit down. Graph the result as the function  $p(x)$ .
  - b. Start with the graph of  $f(x) = x^2$ . Translate it 1 unit down and then reflect it over the  $x$ -axis. Graph the result as the function  $q(x)$ .
  - c. **Construct viable arguments.** Does the order in which the two transformations are performed matter? Explain.
  - d. Write the equations of  $p(x)$  and  $q(x)$ .



My Notes

MATH TERMS

The **vertex form** of a quadratic function is  $f(x) = a(x - h)^2 + k$ , where the vertex of the function is  $(h, k)$ . Notice that the transformations of  $f(x) = x^2$  are apparent when the function is in vertex form.

Learning Targets:

- Write a quadratic function in vertex form.
- Use transformations to graph a quadratic function in vertex form.

**SUGGESTED LEARNING STRATEGIES:** Interactive Word Wall, Marking the Text, Create Representations, Group Presentation, RAFT

A quadratic function in standard form,  $f(x) = ax^2 + bx + c$ , can be changed into **vertex form** by completing the square.

Example A

Write  $f(x) = 3x^2 - 12x + 7$  in vertex form.

**Step 1:** Factor the leading coefficient from the quadratic and linear terms.  $f(x) = 3(x^2 - 4x) + 7$

**Step 2:** Complete the square by taking half the linear coefficient  $[0.5(-4) = -2]$ , squaring it  $[(-2)^2 = 4]$ , and then adding it inside the parentheses.  $f(x) = 3(x^2 - 4x + \overset{\uparrow}{+4}) + 7$

**Step 3:** To maintain the value of the expression, multiply the leading coefficient  $[3]$  by the number added inside the parentheses  $[4]$ . Then subtract that product  $[12]$ .  $f(x) = 3(x^2 - 4x + 4) - 3(4) + 7$   
 $f(x) = 3(x^2 - 4x + 4) - 12 + 7$

**Step 4:** Write the trinomial inside the parentheses as a perfect square. The function is in vertex form.  $f(x) = 3(x - 2)^2 - 5$

**Solution:** The vertex form of  $f(x) = 3x^2 - 12x + 7$  is  $f(x) = 3(x - 2)^2 - 5$ .

Try These A

**Make use of structure.** Write each quadratic function in vertex form. Show your work.

a.  $f(x) = 5x^2 + 40x - 3$

b.  $g(x) = -4x^2 - 12x + 1$

## Lesson 11-3

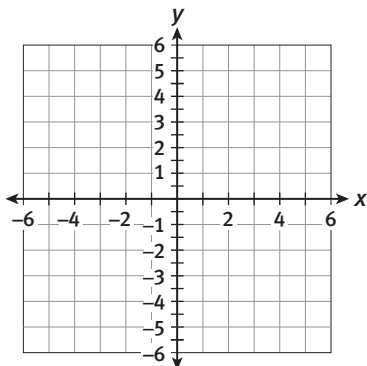
### Vertex Form

## ACTIVITY 11

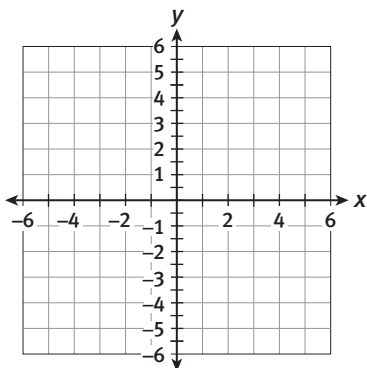
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1. **Make sense of problems.** Write each function in vertex form. Then describe the transformation(s) from the parent function and graph without the use of a graphing calculator.

a.  $f(x) = -2x^2 + 4x + 3$



b.  $g(x) = \frac{1}{2}x^2 + 3x + \frac{3}{2}$



2. Consider the function  $f(x) = 2x^2 - 16x + 34$ .

a. Write the function in vertex form.

b. What is the vertex of the graph of the function? Explain your answer.

My Notes

### MATH TIP

You can check that you wrote the vertex form correctly by rewriting the vertex form in standard form and checking that the rewritten standard form equation matches the original equation.

## My Notes

- c. What is the axis of symmetry of the function's graph? How do you know?
- d. Does the graph of the function open upward or downward? How do you know?

## ACADEMIC VOCABULARY

An **advantage** is a benefit or a desirable feature.

A **disadvantage** is an undesirable feature.

## Check Your Understanding

3. Write a set of instructions for a student who is absent explaining how to write the function  $f(x) = x^2 + 6x + 11$  in vertex form.
4. What are some **advantages** of the vertex form of a quadratic function compared to the standard form?
5. A student is writing  $f(x) = 4x^2 - 8x + 8$  in vertex form. What number should she write in the first box to complete the square inside the parentheses? What number should she write in the second box to keep the expression on the right side of the equation balanced? Explain.

$$f(x) = 4(x^2 - 2x + \square) - \square + 8$$

## LESSON 11-3 PRACTICE

Write each function in vertex form. Then describe the transformation(s) from the parent function and use the transformations to graph the function.

6.  $g(x) = x^2 + 6x + 5$

7.  $h(x) = x^2 - 8x + 17$

8.  $j(x) = 2x^2 + 4x + 5$

9.  $k(x) = -3x^2 + 12x - 7$

Write each function in vertex form. Then identify the vertex and axis of symmetry of the function's graph, and tell which direction the graph opens.

10.  $f(x) = x^2 - 20x + 107$

11.  $f(x) = -x^2 - 16x - 67$

12.  $f(x) = 5x^2 - 20x + 31$

13.  $f(x) = -2x^2 - 12x + 5$

14. **Critique the reasoning of others.** Rebecca says that the function  $f(x) = x^2 - 5$  is written in standard form. Lane says that the function is written in vertex form. Who is correct? Explain.



### ACTIVITY 11 PRACTICE

Write your answers on notebook paper.  
Show your work.

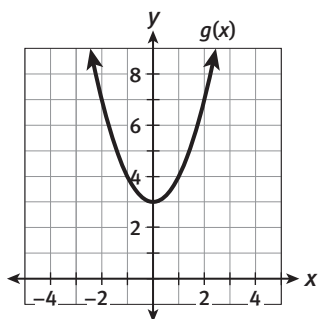
#### Lesson 11-1

For each function, identify all transformations of the function  $f(x) = x^2$ . Then graph the function.

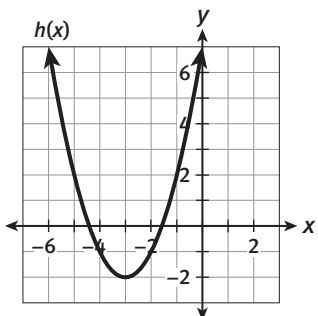
1.  $g(x) = x^2 + 1$
2.  $g(x) = (x - 4)^2$
3.  $g(x) = (x + 2)^2 + 3$
4.  $g(x) = (x - 3)^2 - 4$

Each function graphed below is a translation of  $f(x) = x^2$ . Describe the transformation. Then write the equation of the transformed function.

5.



6.



Use transformations of the parent quadratic function to determine the vertex and axis of symmetry of the graph of each function.

7.  $g(x) = (x - 8)^2$
8.  $g(x) = (x + 6)^2 - 4$

Write a quadratic function  $g(x)$  that represents each transformation of the function  $f(x) = x^2$ .

9. translate 6 units right
10. translate 10 units down
11. translate 9 units right and 6 units up
12. translate 4 units left and 8 units down
13. The function  $g(x)$  is a translation of  $f(x) = x^2$ . The vertex of the graph of  $g(x)$  is  $(-4, 7)$ . What is the equation of  $g(x)$ ? Explain your answer.

#### Lesson 11-2

For each function, identify all transformations of the function  $f(x) = x^2$ . Then graph the function.

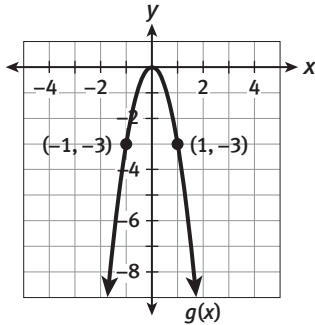
14.  $g(x) = -\frac{1}{3}x^2$
15.  $g(x) = \frac{1}{5}x^2$
16.  $g(x) = \frac{1}{2}(x - 3)^2$
17.  $g(x) = -2(x + 3)^2 + 1$
18.  $g(x) = -3(x + 2)^2 - 5$

Write a quadratic function  $g(x)$  that represents each transformation of the function  $f(x) = x^2$ .

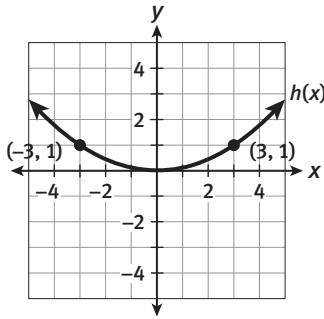
19. shrink horizontally by a factor of  $\frac{1}{4}$
20. stretch vertically by a factor of 8
21. shrink vertically by a factor of  $\frac{1}{3}$ , translate 6 units up
22. translate 1 unit right, stretch vertically by a factor of  $\frac{3}{2}$ , reflect over the  $x$ -axis, translate 7 units up

Each function graphed below is a transformation of  $f(x) = x^2$ . Describe the transformation. Then write the equation of the transformed function.

23.



24.



25. Which of these functions has the widest graph when they are graphed on the same coordinate plane?

- A.  $f(x) = -2x^2$                       B.  $f(x) = 5x^2$   
C.  $f(x) = \frac{1}{2}x^2$                         D.  $f(x) = -\frac{1}{5}x^2$

**Lesson 11-3**

Write each function in vertex form. Then describe the transformation(s) from the parent function and use the transformations to graph the function.

26.  $g(x) = x^2 - 4x - 1$   
27.  $g(x) = -2x^2 + 12x - 17$   
28.  $g(x) = 3x^2 + 6x + 1$

Write each function in vertex form. Then identify the vertex and axis of symmetry of the function's graph, and tell which direction the graph opens.

29.  $f(x) = x^2 - 16x + 71$   
30.  $f(x) = 2x^2 + 36x + 142$   
31.  $f(x) = -3x^2 + 6x + 9$   
32.  $f(x) = x^2 - 2x + 5$   
33. The function  $h(t) = -16t^2 + 22t + 4$  models the height  $h$  in feet of a football  $t$  seconds after it is thrown.  
a. Write the function in vertex form.  
b. To the nearest foot, what is the greatest height that the football reaches? Explain your answer.  
c. To the nearest tenth of a second, how long after the football is thrown does it reach its greatest height? Explain your answer.  
34. Which function has a vertex to the right of the  $y$ -axis?  
A.  $f(x) = -x^2 - 10x - 29$   
B.  $f(x) = x^2 - 12x + 40$   
C.  $f(x) = x^2 + 2x - 5$   
D.  $f(x) = x^2 + 6x + 2$

**MATHEMATICAL PRACTICES**

**Construct Viable Arguments and Critique the Reasoning of Others**

35. A student claims that the function  $g(x) = -x^2 - 5$  has no real zeros. As evidence, she claims that the graph of  $g(x)$  opens downward and its vertex is  $(0, -5)$ , which means that the graph never crosses the  $x$ -axis. Is the student's argument valid? Support your answer.

# Writing and Transforming Quadratic Functions

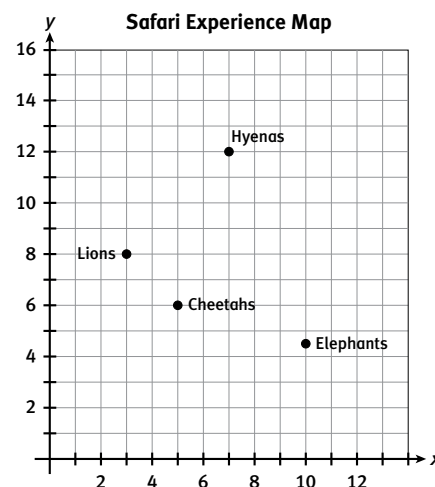
## THE SAFARI EXPERIENCE

## Embedded Assessment 2

Use after Activity 11

A zoo is constructing a new exhibit of African animals called the Safari Experience. A path called the Lion Loop will run through the exhibit. The Lion Loop will have the shape of a parabola and will pass through these points shown on the map: (3, 8) near the lions, (7, 12) near the hyenas, and (10, 4.5) near the elephants.

1. Write the standard form of the quadratic function that passes through the points (3, 8), (7, 12), and (10, 4.5). This function models the Lion Loop on the map.
2. A lemonade stand will be positioned at the vertex of the parabola formed by the Lion Loop.
  - a. Write the equation that models the Lion Loop in vertex form,  $y = a(x - h)^2 + k$ .
  - b. What are the map coordinates of the lemonade stand? Explain how you know.
3. A graphic artist needs to draw the Lion Loop on the map.
  - a. Provide instructions for the artist that describe the shape of the Lion Loop as a set of transformations of the graph of  $f(x) = x^2$ .
  - b. Use the transformations of  $f(x)$  to draw the Lion Loop on the map.
4. The Safari Experience will also have a second path called the Cheetah Curve. This path will also be in the shape of a parabola. It will open to the right and have its focus at the cheetah exhibit at map coordinates (5, 6).
  - a. Choose a vertex for the Cheetah Curve. Explain why the coordinates you chose for the vertex are appropriate.
  - b. Use the focus and the vertex to write the equation that models the Cheetah Curve.
  - c. What are the directrix and the axis of symmetry of the parabola that models the Cheetah Curve?
  - d. Draw and label the Cheetah Curve on the map.



Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
The solution demonstrates these characteristics:				
<p><b>Mathematics Knowledge and Thinking</b> (Items 1, 2, 3a, 4a-c)</p>	<ul style="list-style-type: none"> <li>Effective understanding of quadratic functions as transformations of <math>f(x) = x^2</math></li> <li>Clear and accurate understanding of how to write a quadratic function in standard form given three points on its graph</li> <li>Clear and accurate understanding of how to transform a quadratic function from standard to vertex form</li> <li>Clear and accurate understanding of how to identify key features of a graph of a parabola and how they relate to the equation for a parabola</li> </ul>	<ul style="list-style-type: none"> <li>Adequate understanding of quadratic functions as transformations of <math>f(x) = x^2</math></li> <li>Largely correct understanding of how to write a quadratic function in standard form given three points on its graph</li> <li>Largely correct understanding of how to transform a quadratic function from standard to vertex form</li> <li>Largely correct understanding of how to identify key features of a graph of a parabola and how they relate to the equation for a parabola</li> </ul>	<ul style="list-style-type: none"> <li>Partial understanding of quadratic functions as transformations of <math>f(x) = x^2</math></li> <li>Partial understanding of how to write a quadratic function in standard form given three points on its graph</li> <li>Difficulty with transforming a quadratic function from standard to vertex form</li> <li>Partial understanding of how to identify key features of a graph of a parabola and how they relate to the equation for a parabola</li> </ul>	<ul style="list-style-type: none"> <li>Inaccurate or incomplete understanding of quadratic functions as transformations of <math>f(x) = x^2</math></li> <li>Little or no understanding of how to write a quadratic function in standard form given three points on its graph</li> <li>Little or no understanding of how to transform a quadratic function from standard to vertex form</li> <li>Little or no understanding of how to identify key features of a graph of a parabola and how they relate to the equation for a parabola</li> </ul>
<p><b>Problem Solving</b> (Items 1, 2b, 4b)</p>	<ul style="list-style-type: none"> <li>An appropriate and efficient strategy that results in a correct answer</li> </ul>	<ul style="list-style-type: none"> <li>A strategy that may include unnecessary steps but results in a correct answer</li> </ul>	<ul style="list-style-type: none"> <li>A strategy that results in some incorrect answers</li> </ul>	<ul style="list-style-type: none"> <li>No clear strategy when solving problems</li> </ul>
<p><b>Mathematical Modeling / Representations</b> (Items 1, 2b, 3b, 4b, 4d)</p>	<ul style="list-style-type: none"> <li>Effective understanding of how to model real-world scenarios with quadratic functions and parabolas and interpret their key features</li> <li>Clear and accurate understanding of how to graph quadratic functions using transformations, and how to graph parabolas</li> </ul>	<ul style="list-style-type: none"> <li>Adequate understanding of how to model real-world scenarios with quadratic functions and parabolas and interpret their key features</li> <li>Largely correct understanding of how to graph quadratic functions using transformations, and how to graph parabolas</li> </ul>	<ul style="list-style-type: none"> <li>Partial understanding of how to model real-world scenarios with quadratic functions and parabolas and interpret their key features</li> <li>Some difficulty with understanding how to graph quadratic functions using transformations and with graphing parabolas</li> </ul>	<ul style="list-style-type: none"> <li>Little or no understanding of how to model real-world scenarios with quadratic functions and parabolas and interpret their key features</li> <li>Inaccurate or incomplete understanding of how to graph quadratic functions using transformations, and how to graph parabolas</li> </ul>
<p><b>Reasoning and Communication</b> (Items 2b, 3a, 4a)</p>	<ul style="list-style-type: none"> <li>Precise use of appropriate math terms and language to describe how to graph a quadratic function as a transformation of <math>f(x) = x^2</math></li> <li>Precise use of appropriate math terms and language to explain how features of a graph relate to a real-world scenario</li> </ul>	<ul style="list-style-type: none"> <li>Adequate descriptions of how to graph a quadratic function as a transformation of <math>f(x) = x^2</math></li> <li>Adequate explanation of how features of a graph relate to a real-world scenario</li> </ul>	<ul style="list-style-type: none"> <li>Misleading or confusing descriptions of how to graph a quadratic function as a transformation of <math>f(x) = x^2</math></li> <li>Partially correct explanation of how features of a graph relate to a real-world scenario</li> </ul>	<ul style="list-style-type: none"> <li>Incomplete or inaccurate descriptions of how to graph a quadratic function as a transformation of <math>f(x) = x^2</math></li> <li>Incorrect or incomplete explanation of how features of a graph relate to a real-world scenario</li> </ul>

### Calendar Art

### Lesson 12-1 Key Features of Quadratic Functions

#### Learning Targets:

- Write a quadratic function from a verbal description.
- Identify and interpret key features of the graph of a quadratic function.

**SUGGESTED LEARNING STRATEGIES:** Marking the Text, Paraphrasing, Create Representations, Quickwrite, Self Revision/Peer Revision

Ms. Picasso, sponsor for her school's art club, sells calendars featuring student artwork to raise money for art supplies. A local print shop sponsors the calendar sale and donates the printing and supplies. From past experience, Ms. Picasso knows that she can sell 150 calendars for \$3.00 each. She considers raising the price to try to increase the profit that the club can earn from the sale. However, she realizes that by raising the price, the club will sell fewer than 150 calendars.

1. If Ms. Picasso raises the price of the calendar by  $x$  dollars, write an expression for the price of one calendar.
  
2. In previous years, Ms. Picasso found that for each \$0.40 increase in price, the number of calendars sold decreased by 10. Complete the table below to show that relationship between the price increase and the number of calendars sold.

Increase in price (\$), $x$	Number of calendars sold
0.00	150
0.40	
0.80	
1.20	

3. **Model with mathematics.** Use the data in the table to write an expression that models the number of calendars sold in terms of  $x$ , the price increase.
  
4. Write a function that models  $A(x)$ , the amount of money raised selling calendars when the price is increased  $x$  dollars.

My Notes

#### MATH TIP

If the value of one quantity decreases by a constant amount as another quantity increases by a constant amount, the relationship between the quantities is linear.

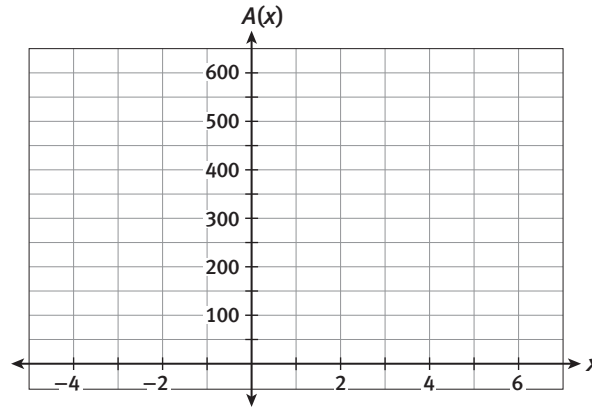
My Notes

**MATH TIP**

A quadratic function in *standard form* is written as  $f(x) = ax^2 + bx + c$ .

5. Write your function  $A(x)$  in standard form. Identify the constants  $a$ ,  $b$ , and  $c$ .

6. Graph  $A(x)$  on the coordinate grid.



7. a. For what values of  $x$  does the value of  $A(x)$  increase as you move from left to right on the graph?

- b. For what values of  $x$  does the value of  $A(x)$  decrease as you move from left to right on the graph?

8. **Reason quantitatively.** Based on the model, what is the maximum amount of money that can be earned? What is the increase in price of a calendar that will yield that maximum amount of money?

## Lesson 12-1

### Key Features of Quadratic Functions

## ACTIVITY 12

continued

9. a. What feature of the graph gives the information that you used to answer Item 8?
- b. How does this feature relate to the intervals of  $x$  for which  $A(x)$  is increasing and decreasing?

The point that represents the maximum value of  $A(x)$  is the *vertex* of this parabola. The  $x$ -coordinate of the vertex of the graph of  $f(x) = ax^2 + bx + c$  can be found using the formula  $x = -\frac{b}{2a}$ .

10. Use this formula to find the  $x$ -coordinate of the vertex of  $A(x)$ .

### MATH TIP

Substitute the  $x$ -coordinate of the vertex into the quadratic equation to find the  $y$ -coordinate of the vertex.

### Check Your Understanding

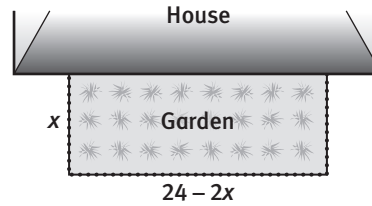
11. Look back at the expression you wrote for  $A(x)$  in Item 4. Explain what each part of the expression equal to  $A(x)$  represents.
12. Is the vertex of the graph of a quadratic function always the highest point? Explain.
13. The graph of a quadratic function  $f(x)$  opens upward, and its vertex is  $(-2, 5)$ . For what values of  $x$  is the value of  $f(x)$  increasing? For what values of  $x$  is the value of  $f(x)$  decreasing? Explain your answers.
14. **Construct viable arguments.** Suppose you are asked to find the vertex of the graph of  $f(x) = -3(x - 4)^2 + 1$ . Which method would you use? Explain why you would choose that method.



My Notes

LESSON 12-1 PRACTICE

Mr. Picasso would like to create a small rectangular vegetable garden adjacent to his house. He has 24 ft of fencing to put around three sides of the garden.



15. **Construct viable arguments.** Explain why  $24 - 2x$  is an appropriate expression for the length of the garden in feet given that the width of the garden is  $x$  ft.
16. Write the standard form of a quadratic function  $G(x)$  that gives the area of the garden in square feet in terms of  $x$ . Then graph  $G(x)$ .
17. What is the vertex of the graph of  $G(x)$ ? What do the coordinates of the vertex represent in this situation?
18. **Reason quantitatively.** What are the dimensions of the garden that yield that maximum area? Explain your answer.

Write each quadratic function in standard form and identify the vertex.

19.  $f(x) = (3x - 6)(x + 4)$

20.  $f(x) = 2(x - 6)(20 - 3x)$



**Learning Targets:**

- Write a quadratic function from a verbal description.
- Identify and interpret key features of the graph of a quadratic function.

**SUGGESTED LEARNING STRATEGIES:** Interactive Word Wall, Quickwrite, Think Aloud, Discussion Groups, Self Revision/Peer Revision

An intercept occurs at the point of intersection of a graph and one of the axes. For a function  $f$ , an  $x$ -intercept is a value  $n$  for which  $f(n) = 0$ . The  $y$ -intercept is the value of  $f(0)$ . Use the graph that you made in Item 6 in the previous lesson for Items 1 and 2 below.

1. What is the  $y$ -intercept of the graph of  $A(x)$ ? What is the significance of the  $y$ -intercept in terms the calendar problem?
  
2. **Make sense of problems.** What are the  $x$ -intercepts of the graph of  $A(x)$ ? What is the significance of each  $x$ -intercept in terms of the calendar problem?
  
3. The  $x$ -intercepts of the graph of  $f(x) = ax^2 + bx + c$  can be found by solving the equation  $ax^2 + bx + c = 0$ . Solve the equation  $A(x) = 0$  to verify the  $x$ -intercepts of the graph.
  
4. **a.** Recall that  $x$  represents the increase in the price of the calendars. Explain what negative values of  $x$  represent in this situation.
  
  
- b.** Recall that  $A(x)$  represents the amount of money raised from selling the calendars. Explain what negative values of  $A(x)$  represent in this situation.

My Notes

**MATH TIP**

As with graphs of linear functions, graphs of quadratic functions have intercepts where the graph intersects one of the axes.

An  **$x$ -intercept** is the  $x$ -coordinate of a point where a graph intersects the  $x$ -axis. Quadratic functions can have 0, 1, or 2  $x$ -intercepts.

A  **$y$ -intercept** is the  $y$ -coordinate of a point where a graph intersects the  $y$ -axis. A quadratic function will only have one  $y$ -intercept.

## My Notes

## MATH TIP

The reasonable domain and range of a function are the values in the domain and range of the function that make sense in a given real-world situation.

## WRITING MATH

You can write a domain of  $4 < x \leq 2$  in interval notation as  $(4, 2]$  and in set notation as  $\{x \mid x \in \mathbb{R}, 4 < x \leq 2\}$ .

## MATH TIP

The vertical line  $x = -\frac{b}{2a}$  is the axis of symmetry for the graph of the function  $f(x) = ax^2 + bx + c$ .

5. a. **Reason quantitatively.** What is a reasonable domain of  $A(x)$ , assuming that the club makes a profit from the calendar sales? Write the domain as an inequality, in interval notation, and in set notation.

b. Explain how you determined the reasonable domain.

6. a. What is a reasonable range of  $A(x)$ , assuming that the club makes a profit from the calendar sales? Write the range as an inequality, in interval notation, and in set notation.

b. Explain how you determined the reasonable range.

7. What is the average of the  $x$ -intercepts in Item 2? How does this relate to the symmetry of a parabola?

**Check Your Understanding**

- 8. Construct viable arguments.** Explain why a quadratic function is an appropriate model for the amount the club will make from selling calendars.
- 9.** Can a function have more than one  $y$ -intercept? Explain.
- 10.** Do all quadratic functions have two  $x$ -intercepts? Explain.
- 11. Reason abstractly.** Explain how the reasonable domain of a quadratic function helps to determine its reasonable range.

**LESSON 12-2 PRACTICE**

Ms. Picasso is also considering having the students in the art club make and sell candles to raise money for supplies. The function  $P(x) = -20x^2 + 320x - 780$  models the profit the club would make by selling the candles for  $x$  dollars each.

- 12.** What is the  $y$ -intercept of the graph of  $P(x)$ , and what is its significance in this situation?
- 13.** What are the  $x$ -intercepts of the graph of  $P(x)$ , and what is their significance in this situation?
- 14.** Give the reasonable domain and range of  $P(x)$ , assuming that the club does not want to lose money by selling the candles. Explain how you determined the reasonable domain and range.
- 15. Make sense of problems.** What selling price for the candles would maximize the club's profit? Explain your answer.

Identify the  $x$ - and  $y$ -intercepts of each function.

**16.**  $f(x) = x^2 + 11x + 30$

**17.**  $f(x) = 4x^2 + 14x - 8$

My Notes

**CONNECT TO TECHNOLOGY**

When answering Items 12–15, it may help you to view a graph of the function on a graphing calculator.

My Notes

**Learning Targets:**

- Identify key features of a quadratic function from an equation written in standard form.
- Use key features to graph a quadratic function.

**SUGGESTED LEARNING STRATEGIES:** Note Taking, Create Representations, Group Presentation, Identify a Subtask, Quickwrite

**Example A**

For the quadratic function  $f(x) = 2x^2 - 9x + 4$ , identify the vertex, the  $y$ -intercept,  $x$ -intercept(s), and the axis of symmetry. Graph the function.

Identify  $a$ ,  $b$ , and  $c$ .

$$a = 2, b = -9, c = 4$$

**Vertex**

Use  $-\frac{b}{2a}$  to find the  $x$ -coordinate of the vertex.

$$-\frac{(-9)}{2(2)} = \frac{9}{4}; f\left(\frac{9}{4}\right) = -\frac{49}{8}$$

Then use  $f\left(-\frac{b}{2a}\right)$  to find the  $y$ -coordinate.

$$\text{vertex: } \left(\frac{9}{4}, -\frac{49}{8}\right)$$

**$y$ -intercept**

Evaluate  $f(x)$  at  $x = 0$ .

$$f(0) = 4, \text{ so } y\text{-intercept is } 4.$$

**$x$ -intercepts**

Let  $f(x) = 0$ .

$$2x^2 - 9x + 4 = 0$$

Then solve for  $x$  by factoring or by using the Quadratic Formula.

$$x = \frac{1}{2} \text{ and } x = 4 \text{ are solutions, so } x\text{-intercepts are } \frac{1}{2} \text{ and } 4.$$

**Axis of Symmetry**

Find the vertical line through the vertex,  $x = -\frac{b}{2a}$ .

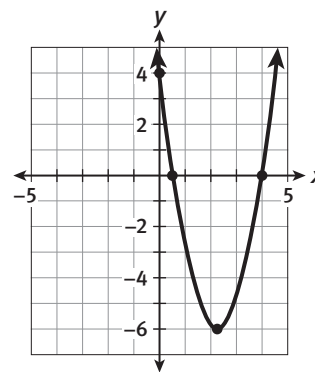
$$x = \frac{9}{4}$$

**Graph**

Graph the points identified above: vertex, point on  $y$ -axis, points on  $x$ -axis.

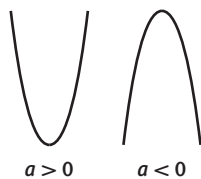
Then draw the smooth curve of a parabola through the points.

The  $y$ -coordinate of the vertex represents the minimum value of the function. The minimum value is  $-\frac{49}{8}$ .



**MATH TIP**

The graph of the function  $f(x) = ax^2 + bx + c$  will open upward if  $a > 0$  and will open downward if  $a < 0$ .



If the parabola opens up, then the  $y$ -coordinate of the vertex is the *minimum* value of the function. If it opens down, the  $y$ -coordinate of the vertex is the *maximum* value of the function.

## Lesson 12-3

### Graphing Quadratic Functions

## ACTIVITY 12

continued

### Try These A

For each quadratic function, identify the vertex, the  $y$ -intercept, the  $x$ -intercept(s), and the axis of symmetry. Then graph the function and classify the vertex as a maximum or minimum.

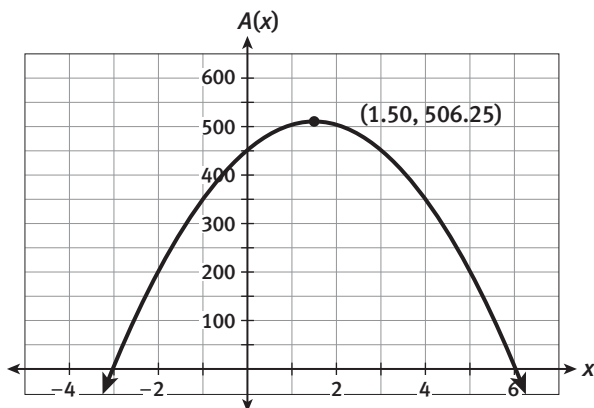
a.  $f(x) = x^2 - 4x - 5$

b.  $f(x) = -3x^2 + 8x + 16$

c.  $f(x) = 2x^2 + 8x + 3$

d.  $f(x) = -x^2 + 4x - 7$

Consider the calendar fund-raising function from Lesson 12-1, Item 5,  $A(x) = -25x^2 + 75x + 450$ , whose graph is below.



- 1. Make sense of problems.** Suppose that Ms. Picasso raises \$450 in the calendar sale. By how much did she increase the price? Explain your answer graphically and algebraically.

My Notes

### MATH TIP

Quadratic equations may be solved by algebraic methods such as factoring or the Quadratic Formula.

An equation can be solved on a graphing calculator by entering each side of the equation as a function, graphing both functions, and finding the points of intersection. The  $x$ -coordinates of the intersection points are the solutions.







My Notes

MATH TIP

The *x*-intercepts of a quadratic function  $y = ax^2 + bx + c$  are the **zeros** of the function. The solutions of a quadratic equation  $ax^2 + bx + c = 0$  are the **roots** of the equation.

Learning Targets:

- Use the discriminant to determine the nature of the solutions of a quadratic equation.
- Use the discriminant to help graph a quadratic function.

**SUGGESTED LEARNING STRATEGIES:** Summarizing, Note Taking, Create Representations, Quickwrite, Self Revision/Peer Revision

The discriminant of a quadratic equation  $ax^2 + bx + c = 0$  can determine not only the nature of the solutions of the equation, but also the number of *x*-intercepts of its related function  $f(x) = ax^2 + bx + c$ .

Discriminant of $ax^2 + bx + c = 0$	Solutions and <i>x</i> -intercepts	Sample Graph of $f(x) = ax^2 + bx + c$
$b^2 - 4ac > 0$  If $b^2 - 4ac$ is: <ul style="list-style-type: none"> <li>● a perfect square</li> <li>● not a perfect square</li> </ul>	<ul style="list-style-type: none"> <li>● Two real solutions</li> <li>● Two <i>x</i>-intercepts</li> <li>● roots are rational</li> <li>● roots are irrational</li> </ul>	<p>A coordinate plane with x and y axes ranging from -5 to 5. A parabola opens upwards, crossing the x-axis at two points: one between -4 and -2, and another between 0 and 2. The vertex is at approximately (-1, -2.5).</p>
$b^2 - 4ac = 0$	<ul style="list-style-type: none"> <li>● One real, rational solution (a double root)</li> <li>● One <i>x</i>-intercept</li> </ul>	<p>A coordinate plane with x and y axes ranging from -5 to 5. A parabola opens upwards, touching the x-axis at exactly one point at x = 2. The vertex is at (2, -1).</p>
$b^2 - 4ac < 0$	<ul style="list-style-type: none"> <li>● Two complex conjugate solutions</li> <li>● No <i>x</i>-intercepts</li> </ul>	<p>A coordinate plane with x and y axes ranging from -5 to 5. A parabola opens upwards, entirely above the x-axis. The vertex is at approximately (2, 1.5).</p>





My Notes

**Learning Targets:**

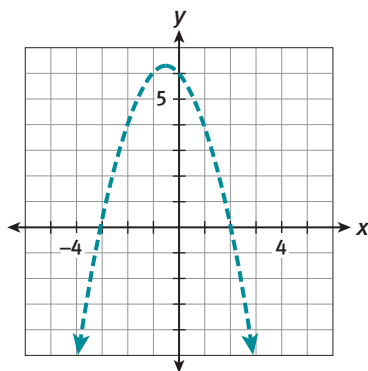
- Graph a quadratic inequality in two variables.
- Determine the solutions to a quadratic inequality by graphing.

**SUGGESTED LEARNING STRATEGIES:** Marking the Text, Create Representations, Guess and Check, Think-Pair-Share, Quickwrite

The solutions to quadratic inequalities of the form  $y > ax^2 + bx + c$  or  $y < ax^2 + bx + c$  can be most easily described using a graph. An important part of solving these inequalities is graphing the related quadratic functions.

**Example A**

Solve  $y > -x^2 - x + 6$ .



Graph the related quadratic function  $y = -x^2 - x + 6$ .

If the inequality symbol is  $>$  or  $<$ , use a *dotted* curve.

If the symbol is  $\geq$  or  $\leq$ , then use a *solid* curve.

This curve divides the plane into two regions.

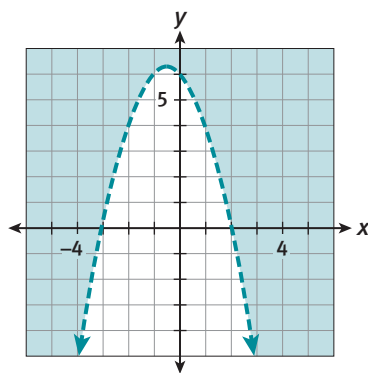
Test  $(0, 0)$  in  $y > -x^2 - x + 6$ .

$$0 > -0^2 - 0 + 6$$

$0 > 6$  is a false statement.

Choose a point on the plane, but not on the curve, to test.

$(0, 0)$  is an easy point to use, if possible.



If the statement is true, shade the region that contains the point. If it is false, shade the other region.

The shaded region represents all solutions to the quadratic inequality.

## Lesson 12-5

### Graphing Quadratic Inequalities

## ACTIVITY 12

continued

### Try These A

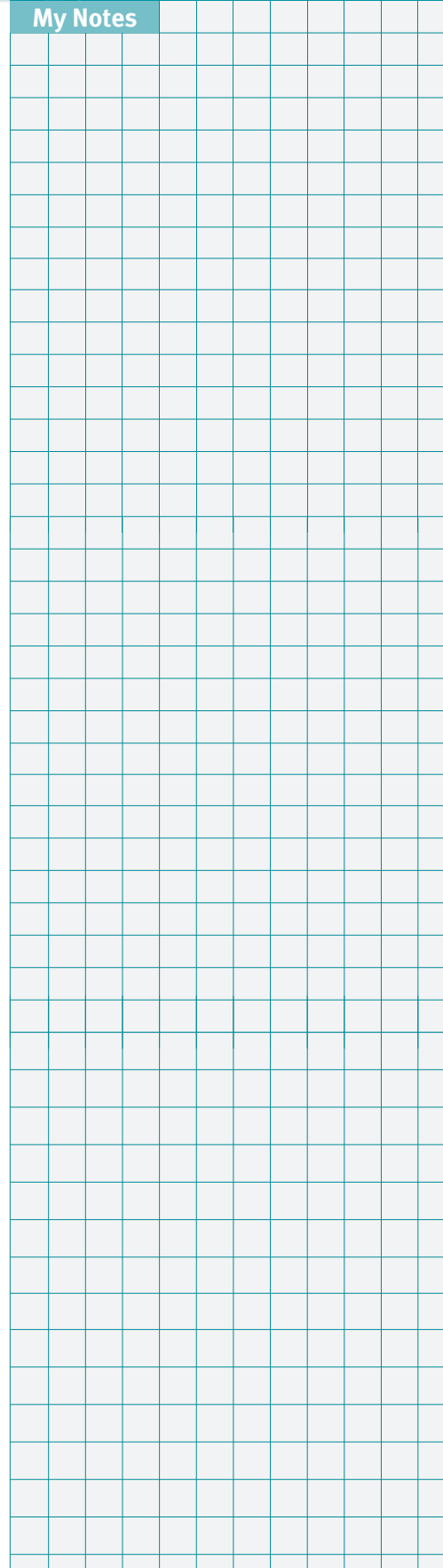
Solve each inequality by graphing.

a.  $y \geq x^2 + 4x - 5$

b.  $y > 2x^2 - 5x - 12$

c.  $y < -3x^2 + 8x + 3$

My Notes



### Check Your Understanding

1. The solutions of which inequality are shown in the graph?

A.  $y \leq -2x^2 + 8x - 7$

B.  $y \geq -2x^2 + 8x - 7$

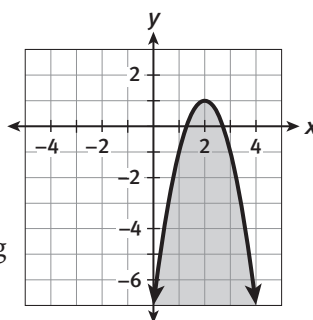
C.  $y \leq 2x^2 - 8x - 7$

D.  $y \geq 2x^2 - 8x - 7$

2. **Reason abstractly.** How does graphing a quadratic inequality in two variables differ from graphing the related quadratic function?

3. Graph the quadratic inequality  $y \geq -x^2 - 6x - 13$ . Then state whether each ordered pair is a solution of the inequality.

a.  $(-1, -6)$     b.  $(-4, -8)$     c.  $(-6, -10)$     d.  $(-2, -5)$



## My Notes

## LESSON 12-5 PRACTICE

Graph each inequality.

4.  $y \leq x^2 + 4x + 7$

5.  $y < x^2 - 6x + 10$

6.  $y > \frac{1}{2}x^2 + 2x + 1$

7.  $y \geq -2x^2 + 4x + 1$

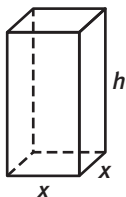
8. **Construct viable arguments.** Give the coordinates of two points that are solutions of the inequality  $y \leq x^2 - 6x + 4$  and the coordinates of two points that are not solutions of the inequality. Explain how you found your answers.

9. **Model with mathematics.** The students in Ms. Picasso's art club decide to sell candles in the shape of square prisms. The height of each candle will be no more than 10 cm. Write an inequality to model the possible volumes in cubic centimeters of a candle with a base side length of  $x$  cm.

10. **Make sense of problems.** Brendan has  $400 \text{ cm}^3$  of wax. Can he make a candle with a base side length of 6 cm that will use all of the wax if the height is limited to 10 cm? Explain your answer using your inequality from Item 9.

## CONNECT TO GEOMETRY

A square prism has two square bases. The volume of a square prism is equal to the area of one of its bases times its height.



### ACTIVITY 12 PRACTICE

Write your answers on notebook paper.

Show your work.

#### Lesson 12-1

The cost of tickets to a whale-watching tour depends on the number of people in the group. For each additional person, the cost per ticket decreases by \$1. For a group with only two people, the cost per ticket is \$44. Use this information for Items 1–7.

- Complete the table below to show the relationship between the number of people in a group and the cost per ticket.

Number of People	Cost per Ticket (\$)
2	
3	
4	
5	

- Use the data in the table to write an expression that models the cost per ticket in terms of  $x$ , the number of people in a group.
- Write a quadratic function in standard form that models  $T(x)$ , the total cost of the tickets for a group with  $x$  people.
- Graph  $T(x)$  on a coordinate grid.
- For what values of  $x$  does the value of  $T(x)$  increase as you move from left to right on the graph?
  - For what values of  $x$  does the value of  $T(x)$  decrease as you move from left to right on the graph?
- What is the vertex of the graph of  $T(x)$ ? What do the coordinates of the vertex represent in this situation?

- Groups on the tour are limited to a maximum size of 20 people. What is the total cost of the tickets for a group of 20 people? Explain how you found your answer.

Write each quadratic function in standard form and identify the vertex.

8.  $f(x) = (4x - 4)(x + 5)$

9.  $f(x) = 4(x + 8)(10 - x)$

#### Lesson 12-2

Mr. Gonzales would like to create a playground in his backyard. He has 20 ft of fencing to enclose the play area. Use this information for Items 10–13.

- Write a quadratic function in standard form that models  $f(x)$ , the total area of the playground in square feet in terms of its width  $x$  in feet. Then graph  $f(x)$ .
- Write the  $x$ - and  $y$ -intercepts of  $f(x)$  and interpret them in terms of the problem.
- Give the reasonable domain and range of  $f(x)$  as inequalities, in interval notation, and in set notation. Explain how you determined the reasonable domain and range.
- What is the maximum area for the playground? What are the dimensions of the playground with the maximum area?

Identify the  $x$ - and  $y$ -intercepts of each function.

14.  $f(x) = x^2 + 3x - 28$

15.  $f(x) = 2x^2 + 13x + 15$

#### Lesson 12-3

For each function, identify the vertex,  $y$ -intercept,  $x$ -intercept(s), and axis of symmetry. Identify whether the function has a maximum or minimum and give its value.

16.  $f(x) = -x^2 + 4x + 5$

17.  $f(x) = 2x^2 - 12x + 13$

18.  $f(x) = -3x^2 + 12x - 9$

19. Explain how to find the  $y$ -intercept of the quadratic function  $f(x) = x^2 - 3x - 18$  without graphing the function.

The function  $h(t) = -5t^2 + 15t + 1$  models the height in meters of an arrow  $t$  seconds after it is shot. Use this information for Items 20 and 21.

20. Based on the model, when will the arrow have a height of 10 m? Round times to the nearest tenth of a second. Explain how you determined your answer.
21. Does the arrow reach a height of 12 m? Justify your answer both graphically and algebraically.

### Lesson 12-4

For each equation, find the value of the discriminant and describe the nature of the solutions. Then find the  $x$ -intercepts.

22.  $2x^2 - 5x - 3 = 0$
23.  $3x^2 + x + 2 = 0$
24.  $4x^2 + 4x + 1 = 0$
25.  $2x^2 + 6x + 3 = 0$
26. A quadratic equation has two distinct rational roots. Which one of the following could be the discriminant of the equation?
- A.  $-6$                       B.  $0$   
C.  $20$                         D.  $64$
27. A quadratic equation has one distinct rational solution. How many  $x$ -intercepts does the graph of the related quadratic function have? Explain your answer.
28. The graph of a quadratic function has no  $x$ -intercepts. What can you conclude about the value of the discriminant of the related quadratic equation? Explain your reasoning.

### Lesson 12-5

Graph each quadratic inequality.

29.  $y < x^2 + 7x + 10$
30.  $y \geq 2x^2 + 4x - 1$
31.  $y > x^2 - 6x + 9$
32.  $y \leq -x^2 + 3x + 4$
33. Which of the following is a solution of the inequality  $y > -x^2 - 8x - 12$ ?
- A.  $(-6, 0)$                       B.  $(-4, -2)$   
C.  $(-3, 1)$                         D.  $(-2, 4)$

The time in minutes a factory needs to make  $x$  cell phone parts in a single day is modeled by the inequality  $y \leq -0.0005x^2 + x + 20$ , for the domain  $0 \leq x \leq 1000$ . Use this information for Items 34–36.

34. a. Is the ordered pair  $(200, 100)$  a solution of the inequality? How do you know?  
b. What does the ordered pair  $(200, 100)$  represent in this situation?
35. What is the longest it will take the factory to make 600 cell phone parts? Explain how you determined your answer.
36. Can the factory complete an order for 300 parts in 4 hours? Explain.
37. Give the coordinates of two points that are solutions of the inequality  $y \leq x^2 - 3x - 10$  and the coordinates of two points that are not solutions of the inequality. Explain how you found your answers.

### MATHEMATICAL PRACTICES

#### Look for and Make Use of Structure

38. Describe the relationship between solving a quadratic equation and graphing the related quadratic function.

### Supply and Demand

### Lesson 13-1 Solving a System Graphically

#### Learning Targets:

- Use graphing to solve a system consisting of a linear and a nonlinear equation.
- Interpret the solutions of a system of equations.

**SUGGESTED LEARNING STRATEGIES:** Close Reading, Think Aloud, Discussion Groups, Create Representations, Look for a Pattern

The owner of Salon Ultra Blue is working with a pricing consultant to determine the best price to charge for a basic haircut. The consultant knows that, in general, as the price of a haircut at a salon goes down, demand for haircuts at the salon goes up. In other words, if Salon Ultra Blue decreases its prices, more customers will want to get their hair cut there.

Based on the consultant's research, customers will demand 250 haircuts per week if the price per haircut is \$20. For each \$5 increase in price, the demand will decrease by 25 haircuts per week.

1. Let the function  $f(x)$  model the quantity of haircuts demanded by customers when the price of haircuts is  $x$  dollars.
  - a. **Reason quantitatively.** What type of function is  $f(x)$ ? How do you know?

- b. Write the equation of  $f(x)$ .

The price of a haircut not only affects demand, but also affects supply. As the price charged for a haircut increases, cutting hair becomes more profitable. More stylists will want to work at the salon, and they will be willing to work longer hours to provide more haircuts.

My Notes

#### CONNECT TO ECONOMICS

In economics, *demand* is the quantity of an item that customers are willing to buy at a particular price. The *law of demand* states that as the price of an item decreases, the demand for the item tends to increase.

#### CONNECT TO ECONOMICS

*Supply* is the quantity of an item that businesses are willing to sell at a particular price. The *law of supply* states that as the price of an item increases, the supply of the item tends to increase.



**My Notes**

The consultant gathered the following data on how the price of haircuts affects the number of haircuts the stylists are willing to supply each week.

**Supply of Haircuts**

Price per Haircut (\$)	Number of Haircuts Available per Week
20	15
30	55
40	115
50	195

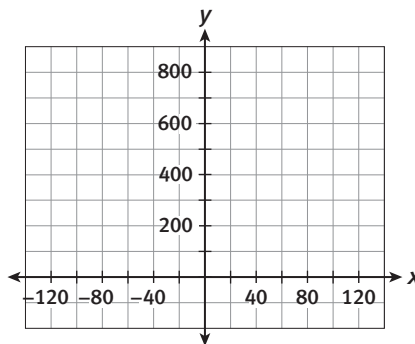
**CONNECT TO TECHNOLOGY**

One way to write the equation of the quadratic function is to perform a quadratic regression on the data in the table. See Activity 10 for more information.

- The relationship shown in the table is quadratic. Write the equation of a quadratic function  $g(x)$  that models the quantity of haircuts the stylists are willing to supply when the price of haircuts is  $x$  dollars.

- Model with mathematics.** Write a system of two equations in two variables for the demand and supply functions. In each equation, let  $y$  represent the quantity of haircuts and  $x$  represent the price in dollars per haircut.

- Graph the system on the coordinate plane.





## Lesson 13-1

### Solving a System Graphically

## ACTIVITY 13

continued

5. Explain how you determine the location of the solutions on the graph in Item 4.
6. Explain the relationship of the solution to the demand function  $f(x)$  and the supply function  $g(x)$ .
7. Use the graph to approximate the solutions of the system of equations.

Now use a graphing calculator to make better approximations of the solutions of the system of equations. First, enter the equations from the system as Y1 and Y2.

8. **Use appropriate tools strategically.** Now view a table showing values of X, Y1, and Y2.
  - a. How can you approximate solutions of a system of two equations in two variables by using a table of values on a graphing calculator?
  - b. Use the table to approximate the solutions of the system. Find the coordinates of the solutions to the nearest integer.

### My Notes

### TECHNOLOGY TIP

You can change the table settings on a graphing calculator by pressing  $\boxed{2nd}$  and then the key with TblSet printed above it. The table start setting (TblStart) lets you change the first value of X displayed in the table. The table step setting ( $\Delta$ Tbl) lets you adjust the change in X between rows of the table.

## My Notes

## TECHNOLOGY TIP

To use the intersect feature on a graphing calculator, access the calculate menu by pressing  $\boxed{2\text{nd}}$  and then the key with Calc printed above it. Next, select **5: Intersect**, and then follow the instructions.

9. Next, view a graph of the system of equations on the graphing calculator. Adjust the viewing window as needed so that the intersection points of the graphs of the equations are visible. Then use the intersect feature to approximate the solutions of the system of equations.
  
10. Explain why one of the solutions you found in Item 9 does not make sense in the context of the supply and demand functions for haircuts at the salon.
  
11. **Make sense of problems.** Interpret the remaining solution in the context of the situation.
  
12. Explain why the solution you described in Item 11 is reasonable.
  
13. The pricing consultant recommends that Salon Ultra Blue price its haircuts so that the weekly demand is equal to the weekly supply. Based on this recommendation, how much should the salon charge for a basic haircut?



## My Notes

## ACADEMIC VOCABULARY

A **counterexample** is an example that demonstrates that a statement is not true.

## Check Your Understanding

- When interpreting the solution of the system in Item 11, how did you decide how to round the  $x$ - and  $y$ -coordinates of the solution?
- Construct viable arguments.** Can a system of a linear equation and a quadratic equation have infinitely many solutions? Explain your reasoning.
- A student claims that if a system of a linear equation and a quadratic equation has two real solutions, then a graph of the system will have one intersection point to the left of the vertex of the parabola and one intersection point to the right of the vertex. Provide a **counterexample** to show that the student's claim is not correct.
- Compare and contrast using a graph and a table to approximate the solution of a system of one linear equation and one quadratic equation.

## LESSON 13-1 PRACTICE

The owner of Salon Ultra Blue also wants to set the price for styling hair for weddings, proms, and other formal events.

- Make sense of problems.** Based on the pricing consultant's research, customers will demand 34 formal hairstyles per week if the price per hairstyle is \$40. For each \$10 increase in price, the demand will decrease by 4 hairstyles per week. Write a linear function  $f(x)$  that models the quantity of formal hairstyles demanded by customers when the price of the hairstyles is  $x$  dollars.

- The table shows how the price of formal hairstyles affects the number the stylists are willing to supply each week. Write the equation of a quadratic function  $g(x)$  that models the quantity of formal hairstyles the stylists are willing to supply when the price of hairstyles is  $x$  dollars.

Supply of Formal Hairstyles

Price per Hairstyle (\$)	Number Available per Week
40	3
50	9
60	17

- Model with mathematics.** Write a system of two equations in two variables for the demand and supply functions. In each equation, let  $y$  represent the quantity of formal hairstyles and  $x$  represent the price in dollars per hairstyle.
- Approximate the solutions of the system by using a graph or table.
- How much should the salon charge for a formal hairstyle so that the weekly demand is equal to the weekly supply? Explain how you determined your answer.
- Explain why your answer to Item 24 is reasonable.

**Learning Targets:**

- Use substitution to solve a system consisting of a linear and nonlinear equation.
- Determine when a system consisting of a linear and nonlinear equation has no solution.

**SUGGESTED LEARNING STRATEGIES:** Summarizing, Identify a Subtask, Think-Pair-Share, Drafting, Self Revision/Peer Revision

In the last lesson, you approximated the solutions to systems of one linear equation and one quadratic equation by using tables and graphs. You can also solve such systems algebraically, just as you did when solving systems of two linear equations.

**Example A**

The following system represents the supply and demand functions for basic haircuts at Salon Ultra Blue, where  $y$  is the quantity of haircuts demanded or supplied when the price of haircuts is  $x$  dollars. Solve this system algebraically to find the price at which the supply of haircuts equals the demand.

$$\begin{cases} y = -5x + 350 \\ y = \frac{1}{10}x^2 - x - 5 \end{cases}$$

**Step 1:** Use substitution to solve for  $x$ .

$$\begin{aligned} y &= -5x + 350 \\ -5x + 350 &= \frac{1}{10}x^2 - x - 5 \\ 0 &= \frac{1}{10}x^2 + 4x - 355 \\ 0 &= x^2 + 40x - 3550 \\ x &= \frac{-40 \pm \sqrt{40^2 - 4(1)(-3550)}}{2(1)} \\ x &= -20 \pm 5\sqrt{158} \\ x &\approx -82.85 \text{ or } x \approx 42.85 \end{aligned}$$

The first equation is solved for  $y$ .  
Substitute for  $y$  in the second equation.  
Write the equation in standard form.  
Multiply both sides by 10 to eliminate the fraction.

Use the Quadratic Formula.

**Step 2:** Substitute each value of  $x$  into one of the original equations to find the corresponding value of  $y$ .

$$\begin{aligned} y &= -5x + 350 & y &= -5x + 350 \\ y &\approx -5(-82.85) + 350 & y &\approx -5(42.85) + 350 \\ y &\approx 764 & y &\approx 136 \end{aligned}$$

**My Notes**

**MATH TIP**

In this example, the exact values of  $x$  are irrational. Because  $x$  represents a price in dollars, use a calculator to find rational approximations of  $x$  to two decimal places.

## My Notes

**Step 3:** Write the solutions as ordered pairs.

The solutions are approximately  $(-82.85, 764)$  and  $(42.85, 136)$ . Ignore the first solution because a negative value of  $x$  does not make sense in this situation.

**Solution:** The price at which the supply of haircuts equals the demand is \$42.85. At this price, customers will demand 136 haircuts, and the stylists will supply them.

**Try These A**

Solve each system algebraically. Check your answers by substituting each solution into one of the original equations. Show your work.

a. 
$$\begin{cases} y = -2x - 7 \\ y = -2x^2 + 4x + 1 \end{cases}$$

b. 
$$\begin{cases} y = x^2 + 6x + 5 \\ y = 2x + 1 \end{cases}$$

c. 
$$\begin{cases} y = \frac{1}{2}(x + 4)^2 + 5 \\ y = \frac{17}{2} - x \end{cases}$$

d. 
$$\begin{cases} y = -4x^2 + 5x - 8 \\ y = -3x - 24 \end{cases}$$

1. Use substitution to solve the following system of equations. Show your work.

$$\begin{cases} y = 4x + 24 \\ y = -x^2 + 18x - 29 \end{cases}$$

## Lesson 13-2

### Solving a System Algebraically

## ACTIVITY 13

continued

- Describe the solutions of the system of equations from Item 1.
- Use appropriate tools strategically.** Confirm that the system of equations from Item 1 has no real solutions by graphing the system on a graphing calculator. How does the graph show that the system has no real solutions?

### Check Your Understanding

- How does solving a system of one linear equation and one quadratic equation by substitution differ from solving a system of two linear equations by substitution?
- Reason abstractly.** What is an advantage of solving a system of one linear equation and one quadratic equation algebraically rather than by graphing or using a table of values?
- Write a journal entry in which you explain step by step how to solve the following system by using substitution.

$$\begin{cases} y = 2x^2 - 3x + 6 \\ y = -2x + 9 \end{cases}$$

- Could you solve the system in Item 6 by using elimination rather than substitution? Explain.
- Explain how you could use the discriminant of a quadratic equation to determine how many real solutions the following system has.

$$\begin{cases} y = 4x - 21 \\ y = x^2 - 4x - 5 \end{cases}$$

### My Notes

### MATH TIP

To review solving a system of equations by elimination, see Activity 3.

## My Notes

## LESSON 13-2 PRACTICE

Find the real solutions of each system algebraically. Show your work.

$$9. \begin{cases} y = -3x - 8 \\ y = x^2 + 4x + 2 \end{cases}$$

$$10. \begin{cases} y = -2x^2 + 16x - 26 \\ y = 72 - 12x \end{cases}$$

$$11. \begin{cases} y = \frac{1}{4}x^2 - 6x + 1 \\ y = \frac{3}{4}x - \frac{23}{2} \end{cases}$$

$$12. \begin{cases} y = (x - 5)^2 - 3 \\ y = -2x - 3 \end{cases}$$

The owner of Salon Ultra Blue is setting the price for hair highlights. The following system represents the demand and supply functions for hair highlights, where  $y$  is the quantity demanded or supplied per week for a given price  $x$  in dollars.

$$\begin{cases} y = -0.8x + 128 \\ y = 0.03x^2 - 1.5x + 18 \end{cases}$$

13. Use substitution to solve the system of equations.
14. **Attend to precision.** How much should the salon charge for hair highlights so that the weekly demand is equal to the weekly supply? Explain how you determined your answer.
15. Explain why your answer to Item 14 is reasonable.



**ACTIVITY 13 PRACTICE**

Write your answers on notebook paper.  
Show your work.

**Lesson 13-1**

Lori was partway up an escalator when her friend Evie realized that she had Lori's keys. Evie, who was still on the ground floor, tossed the keys up to Lori. The function  $f(x) = -16x^2 + 25x + 5$  models the height in feet of the keys  $x$  seconds after they were thrown. Use this information for Items 1–5.

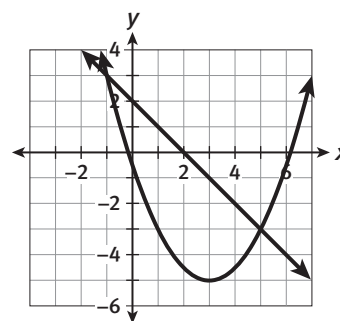
- When the keys are thrown, Lori's hands are 9 ft above ground level and moving upward at a rate of 0.75 ft/s. Write the equation of a function  $g(x)$  that gives the height of Lori's hands compared to ground level  $x$  seconds after the keys are thrown.
- Write the functions  $f(x)$  and  $g(x)$  as a system of two equations in two variables. In each equation, let  $y$  represent height in feet and  $x$  represent time in seconds.
- Graph the system of equations, and use the graph to approximate the solutions of the system.
- How long after the keys are thrown will Lori be able to catch them? Assume that Lori can catch the keys when they are at the same height as her hands. Explain how you determined your answer.
- Explain why your answer to Item 4 is reasonable.

Solve each system by using a graph or table (answers will be approximate).

- $$\begin{cases} y = 10 - 2x \\ y = x^2 - 12x + 31 \end{cases}$$
- $$\begin{cases} y = 5x + 39 \\ y = x^2 + 14x + 52 \end{cases}$$
- $$\begin{cases} y = -2(x - 3)^2 + 9 \\ y = -4x + 3 \end{cases}$$

Use a graph to determine the number of real solutions of each system.

- $$\begin{cases} y = 3x^2 + 6x + 4 \\ y = 0.5x + 8 \end{cases}$$
- $$\begin{cases} y = -2x^2 + 8x - 10 \\ y = -2x + 4 \end{cases}$$
- $$\begin{cases} y = 24 - 4x \\ y = x^2 - 12x + 40 \end{cases}$$
- Which ordered pair is a solution of the system of equations graphed below?



- A.  $(-3, 5)$                       B.  $(-1, 3)$   
C.  $(2, 0)$                          D.  $(3, -5)$

A parallelogram has a height of  $x$  cm. The length of its base is 4 cm greater than its height. A triangle has the same height as the parallelogram. The length of the triangle's base is 20 cm.

- Write a system of two equations in two variables that can be used to determine the values of  $x$  for which the parallelogram and the triangle have the same area.
- Solve the system by using a graph or table.
- Interpret the solutions of the system in the context of the situation.

**Lesson 13-2**

Solve each system algebraically. Check your answers by substituting each solution into one of the original equations. Show your work.

$$16. \begin{cases} y = x - 7 \\ y = -x^2 - 2x - 7 \end{cases}$$

$$17. \begin{cases} y = 2x^2 - 12x + 26 \\ y = 8x - 24 \end{cases}$$

$$18. \begin{cases} y = -3(x - 4)^2 + 2 \\ y = 6x - 31 \end{cases}$$

$$19. \begin{cases} y = -0.5x - 1 \\ y = 0.5x^2 + 3x - 5 \end{cases}$$

A map of a harbor is laid out on a coordinate grid, with the origin marking a buoy at the center of the harbor. A fishing boat is following a path that can be represented on the map by the equation  $y = x^2 - 2x - 4$ . A ferry is following a linear path that passes through the points  $(-3, 7)$  and  $(0, -5)$  when represented on the map. Use this information for Items 20–22.

20. Write a system of equations that can be used to determine whether the paths of the boats will cross.
21. Use substitution to solve the system.
22. Interpret the solution(s) of the system in the context of the situation.
23. How many real solutions does the following system have?

$$\begin{cases} y = -x^2 + 4x \\ y = 3x + 5 \end{cases}$$

- A. none                      B. one  
C. two                        D. infinitely many
24. Explain how you can support your answer to Item 23 algebraically.

A picture-framing company sells two types of glass: regular and nonglare. For a piece of nonglare glass, the charge is equal to the length of the longest side in inches multiplied by the rate \$0.75 per inch. The table shows the charge for several sizes of regular glass.

Charge for Regular Glass

Length of Longest Side (in.)	Charge (\$)
12	3.96
18	7.56
24	12.24

25. Write a linear function  $f(x)$  that gives the charge in dollars for a piece of nonglare glass whose longest side measures  $x$  inches.
26. Write a quadratic function  $g(x)$  that gives the charge in dollars for a piece of regular glass whose longest side measures  $x$  inches.
27. Write the functions  $f(x)$  and  $g(x)$  as a system of equations in terms of  $y$ , the charge in dollars for a piece of glass, and  $x$ , the length of the longest side in inches.
28. Solve the system by using substitution.
29. For what length will the charge for nonglare glass be the same as the charge for regular glass? What will the charge be? Explain your answers.

**MATHEMATICAL PRACTICES****Reason Abstractly and Quantitatively**

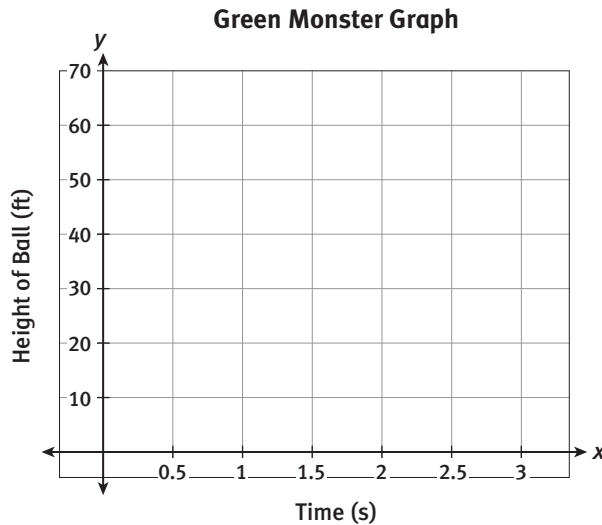
30. Austin sells sets of magnets online. His cost in dollars of making the magnets is given by  $f(x) = 200 + 8x - 0.01x^2$ , where  $x$  is the number of magnet sets he makes. His income in dollars from selling the magnets is given by  $g(x) = 18x$ , where  $x$  is the number of magnet sets he sells. Write and solve the system, and then explain what the solution(s) mean in the context of the situation.

# Graphing Quadratic Functions and Solving Systems

## THE GREEN MONSTER

During a Boston Red Sox baseball game at Fenway Park, the opposing team hit a home run over the left field wall. An unhappy Red Sox fan caught the ball and threw it back onto the field. The height of the ball,  $h(t)$ , in feet,  $t$  seconds after the fan threw the baseball, is given by the function  $h(t) = -16t^2 + 32t + 48$ .

- Graph the equation on the coordinate grid below.



### CONNECT TO HISTORY

The left field wall in Fenway Park is called the Green Monster, a reference to its unusual height.

- Find each measurement value described below. Then tell how each value relates to the graph.
  - At what height was the fan when he threw the ball?
  - What was the maximum height of the ball after the fan threw it?
  - When did the ball hit the field?
- What are the reasonable domain and reasonable range of  $h(t)$ ? Explain how you determined your answers.
- Does the baseball reach a height of 65 ft? Explain your answer both graphically and algebraically.
- Each baseball team in a minor league plays each other team three times during the regular season.

- The table shows the relationship between the number of teams in a baseball league and the total number of games required for each team to play each of the other teams three times.

Number of Teams, $x$	Number of Games, $y$
2	3
3	9
4	18
5	30

- Write a quadratic equation that models the data in the table.
- Last season, the total number of games played in the regular season was 35 more than 10 times the number of teams. Use this information to write a linear equation that gives the number of regular games  $y$  in terms of the number of teams  $x$ .
  - Write a system of equations using the quadratic equation from part a and the linear equation from part b. Then solve the system and interpret the solutions.

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
<b>Mathematics Knowledge and Thinking</b> (Items 2, 4, 5)	The solution demonstrates these characteristics:			
	<ul style="list-style-type: none"> <li>Effective understanding of how to solve quadratic equations and systems of equations</li> <li>Clear and accurate understanding of how to write linear and quadratic models from verbal descriptions or tables of data</li> <li>Clear and accurate understanding of how to use an equation or graph to identify key features of a quadratic function</li> </ul>	<ul style="list-style-type: none"> <li>Adequate understanding of how to solve quadratic equations and systems of equations</li> <li>Largely correct understanding of how to write linear and quadratic models from verbal descriptions or tables of data</li> <li>Largely correct understanding of how to use an equation or graph to identify key features of a quadratic function</li> </ul>	<ul style="list-style-type: none"> <li>Partial understanding of how to solve quadratic equations and systems of equations</li> <li>Partial understanding of how to write linear and quadratic models from verbal descriptions or tables of data</li> <li>Difficulty with using an equation or graph to identify key features of a quadratic function</li> </ul>	<ul style="list-style-type: none"> <li>Inaccurate or incomplete understanding of how to solve quadratic equations and systems of equations</li> <li>Little or no understanding of how to write linear and quadratic models from verbal descriptions or tables of data</li> <li>Little or no understanding of how to use an equation or graph to identify key features of a quadratic function</li> </ul>
<b>Problem Solving</b> (Items 2, 4, 5c)	<ul style="list-style-type: none"> <li>An appropriate and efficient strategy that results in a correct answer</li> </ul>	<ul style="list-style-type: none"> <li>A strategy that may include unnecessary steps but results in a correct answer</li> </ul>	<ul style="list-style-type: none"> <li>A strategy that results in some incorrect answers</li> </ul>	<ul style="list-style-type: none"> <li>No clear strategy when solving problems</li> </ul>
<b>Mathematical Modeling / Representations</b> (Items 1, 2, 3, 4, 5)	<ul style="list-style-type: none"> <li>Effective understanding of how to interpret solutions to a system of equations that represents a real-world scenario</li> <li>Clear and accurate understanding of how to model real-world scenarios with quadratic and linear functions, including reasonable domain and range</li> <li>Clear and accurate understanding of how to graph and interpret key features of a quadratic function that represents a real-world scenario</li> </ul>	<ul style="list-style-type: none"> <li>Adequate understanding of how to interpret solutions to a system of equations that represents a real-world scenario</li> <li>Largely correct understanding of how to model real-world scenarios with quadratic and linear functions, including reasonable domain and range</li> <li>Largely correct understanding of how to graph and interpret key features of a quadratic function that represents a real-world scenario</li> </ul>	<ul style="list-style-type: none"> <li>Partial understanding of how to interpret solutions to a system of equations that represents a real-world scenario</li> <li>Some difficulty with modeling real-world scenarios with quadratic and linear functions, including reasonable domain and range</li> <li>Some difficulty with graphing and interpreting key features of a quadratic function that represents a real-world scenario</li> </ul>	<ul style="list-style-type: none"> <li>Little or no understanding of how to interpret solutions to a system of equations that represents a real-world scenario</li> <li>Inaccurate or incomplete understanding of how to model real-world scenarios with quadratic and linear functions, including reasonable domain and range</li> <li>Inaccurate or incomplete understanding of how to graph and interpret key features of a quadratic function that represents a real-world scenario</li> </ul>
<b>Reasoning and Communication</b> (Items 2, 3, 4)	<ul style="list-style-type: none"> <li>Precise use of appropriate math terms and language to relate the features of a quadratic model, including reasonable domain and range, to a real-world scenario</li> <li>Clear and accurate use of mathematical work to explain whether or not the height could reach 65 feet</li> </ul>	<ul style="list-style-type: none"> <li>Adequate explanations to relate the features of a quadratic model, including reasonable domain and range, to a real-world scenario</li> <li>Correct use of mathematical work to explain whether or not the height could reach 65 feet</li> </ul>	<ul style="list-style-type: none"> <li>Misleading or confusing explanations to relate the features of a quadratic model, including reasonable domain and range, to a real-world scenario</li> <li>Partially correct explanation of whether or not the height could reach 65 feet</li> </ul>	<ul style="list-style-type: none"> <li>Incomplete or inaccurate explanations to relate the features of a quadratic model, including reasonable domain and range, to a real-world scenario</li> <li>Incorrect or incomplete explanation of whether or not the height could reach 65 feet</li> </ul>